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FRAMED STRUCTURES AND GIRDERS

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FRAMED STRUCTURES AND GIRDERS

THEORY AND PRACTICE

VOLUME I

STRESSES

PART I

BY

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PREFACE

This work has been planned to appear in three volumes. The present volume is devoted mainly to the derivation of the fundamental principles of statics, the determination of shears and bending moments in beams, and the analysis of roof and bridge trusses resting on two supports. The second volume will contain an elementary treatment of continuous girders, cantilever, suspension, arch and movable bridges, and will also deal briefly with secondary stresses and the analysis of trusses with redundant members. The third volume will be devoted to the subjects of general design, details of construction, and erection.

While the work is designed to serve primarily as a text-book and its scope and form of treatment will be determined chiefly with a view to that purpose, it is hoped that it will also be found useful as a reference book. The numerous cross-references in the present volume are calculated to facilitate greatly its use by the occasional reader for reference purposes.

In the preparation of this first volume the author has endeavored to meet certain desiderata suggested by a somewhat extended experience both in structural engineering practice and in teaching. Instead of assuming that the student commands an adequate working knowledge of the principles of statics especially applicable to this subject—an assumption rarely justified—some of the earlier chapters are devoted partly or wholly to a brief but critical consideration of those principles. Similarly, on account of their fundamental importance, the general principles governing shears and bending moments in beams are reviewed at the beginning of Chapter IV.

The general methods—both graphic and algebraic—applicable to the analysis of stresses in framed structures are first presented in Chapter V. To do so advantageously both roof and bridge trusses are treated in this chapter. In Chapter VI, which is devoted exclusively to the stresses in roof trusses, it is sought to bring out important underlying principles, rather than to multiply examples involving no new principles. For collateral drill at this point the student's drawing board should be brought into requisition freely.

For purposes of stress analysis, bridge trusses are divided into three general classes, namely, trusses with horizontal chords, trusses with inclined chords, and trusses with sub-panels. From a teaching standpoint this grouping is believed to offer marked advantages over a classification based on the character of the loading. For each class of trusses the loading is considered in the order of (*a*) dead load, (*b*) uniform live load by the conventional method, (*c*) concentrated live loads, and (*d*) uniform live load by the exact method.

The analysis of bridge trusses is presented first mainly by algebraic methods, graphic methods being interspersed only to the limited extent that seemed judicious at this point. A more complete treatment of graphic methods in their application to live-load stresses is reserved for a later chapter devoted wholly to a discussion of the load and moment chart and the treatment of influence lines.

The analysis of the stresses in bridge trusses with sub-panels in Chapter X is developed to an extent commensurate with its importance and its difficulty to the student. No efforts have been spared to present this matter in the simplest and clearest manner consistent with rigor and completeness of treatment.

A chapter devoted to a brief historic account of bridge building and the evolution of modern bridge trusses, with special reference to conditions in America, is introduced as soon as the student's preparation may be expected to enable him to follow its contents intelligently and with interest. This statement applies also to the chapter on live loading, which is partly of a historic nature.

The subjects of skew bridges and bridges on curves, as well as the stresses in lateral and sway bracing and in viaduct towers, are all presented with more than ordinary fullness, but nevertheless as concisely as seemed compatible with a fairly adequate treatment of these important subjects, in the light of practical requirements. The volume concludes with a chapter dealing briefly with the deformation of trusses and camber.

It has been the constant endeavor of the author to present each demonstration in the simplest and most direct form consistent with accuracy and completeness of treatment; to make the illustrations as clear and as nearly self-explanatory as possible; and to employ a simple and consistent notation in general accord with the best current usage.

Under the severe time limitations to which engineering courses are commonly subject, it is hardly to be expected that the entire book should be covered in class-work, nor is it at all necessary to do so. The experienced teacher will recognize at once that many parts of the book may be omitted without seriously impairing the continuity of the subject. In fact in many instances throughout the book the same problem is solved by two or more methods with a view to breadth and variety of treatment, such parts readily lending themselves to abridgment. In general, care has been taken to lay special emphasis on those methods which by reason of superior merit have gained widest acceptance in engineering circles, and on the other hand to avoid purely academic methods totally unsuited to the requirements of engineering practice.

The moment table of Cooper's E-60 Loading, of which two copies accompany each book, will commend itself, it is believed, as a helpful feature in the solution of problems.

The author desires to acknowledge his indebtedness to two of his former pupils for many valuable suggestions. The entire volume was read critically in proof by Mr. F. P. Witmer, Engineer in charge of Bridge Design with the American Bridge Company, and both in manuscript and proof by Mr. H. W. Troelsch, Instructor in Civil Engineering in the University of Pennsylvania.

About two-thirds of the book (308 pp.) have been used in the form of preprints by the author's classes. These pages, having been carefully revised, and in small part rewritten, in the light of class-room experience, may be regarded as virtually in their second edition.

E. M.

PHILADELPHIA, PA., September, 1911.

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FRAMED STRUCTURES AND GIRDERS

THEORY AND PRACTICE

CHAPTER I

GENERAL CONSIDERATIONS AND DEFINITIONS

1. Statics is that branch of mechanics which, in its application to structures in general, concerns itself with the conditions under which every part of a structure, and therefore the structure as a whole, is in equilibrium, i.e., at rest, when the structure is subjected to the action of external forces.

2. The External Forces are the loads to which a structure is subjected (including its own weight) and the reactions developed at the supports under the action of these loads. In order that the structure as a whole may be in equilibrium the external forces must be in equilibrium. In American practice the external forces are expressed in pounds, thousands of pounds, or tons.

A load is said to be *concentrated* or *distributed* according to whether it is applied at a point or distributed over a surface. The load per unit of area is called the *intensity* of the load. If the load on every part of a surface is proportional to the area of that part, the load is said to be *uniformly distributed*, or of *uniform intensity*.

3. The Internal Forces, or Stresses, are the forces developed within the different parts of the structure under the action of the external forces. In order that any given part of a structure may be in equilibrium the external forces (if any) on that part, together with the stresses at all points of junction between that part and the remainder of the structure, must be in equilibrium.

The stresses at the points of junction may be regarded as external forces with respect to the part in question. Stresses are expressed in pounds, thousands of pounds, or tons.

Unit Stress. In a prismatic piece the imaginary line traversing the center of gravity of every cross-section of the piece is called its longitudinal axis, or simply its *axis*. The stress in a prismatic bar is said to be 'axial' when the resultant of the stresses at every cross-section traverses the center of gravity of that section. Axial stresses are usually uniformly distributed, or nearly so, over every cross-section of the piece. The *unit stress*, or the *intensity* of the stress, is then the total stress, S, at the cross-section divided by the area, a, of the cross-section. If s denotes the unit stress, $s=S/a$. Unit stresses are expressed in pounds, or thousands of pounds, per square inch.

4. Strain. Since no material is perfectly rigid, the external forces acting upon a structure produce deformations, called *strains*, in the different parts of the structure. The cohesion of the particles of which the material is composed tends to resist such strains, and this action gives rise to the internal forces, i.e., stresses. The strains increase and the stresses likewise until the latter are sufficiently great to hold the external forces in equilibrium.

The deformation of a prismatic piece under axial stress for any part of its original length is called its strain for that length, and this strain, Δ, divided by the original length, l, within which it is produced, is called the *unit strain*. If δ denotes the unit strain, $\delta=\Delta/l$. The quantities Δ and l are usually expressed in inches, but since the same unit of measurement must be used for both, δ is an abstract number, and its value is therefore independent of the unit of measurement.

The unit strain may also be regarded as the deformation per unit of original length. Since every unit of original length is deformed in proportion to the length of that unit, the unit strain as thus defined is also seen to be independent of the unit of measurement employed.

The stress is tensile or compressive according to whether the external forces cause a piece to lengthen or shorten. If the stress is tensile, the stress as well as the strain is called *positive*, and distinguished by the sign +. Similarly, a compressive stress and the accompanying strain is called *negative*, and distinguished by the sign −. These signs are easily remembered from the fact that under *plus* strain and stress the length of a piece is *increased,* and under a *minus* strain and stress the length is *decreased.*

5. The Ultimate Strength of a material is the maximum unit stress it is capable of developing before failure. Its value is found by dividing the maximum total stress by the original area of the cross-section—that is to say, the effect of the lateral strains, induced by axial loads, on the dimensions of the cross-section is ignored.

The ultimate tensile strength of steel of the grade commonly used for structural purposes ranges usually from 55,000 to 65,000 lbs. per square inch. Such steel will hereafter be referred to briefly as 'structural' steel. The ultimate tensile strength of carbon steel in general may range from about 45,000 lbs. per square inch, for 'extra-soft' rivet steel, to over 400,000 lbs. per square inch for 'cold-drawn' wire.

6. Elastic Limit. Experiments have shown that until the unit stress has reached a certain limit, called the *elastic limit,* it is approximately true for all materials of construction, that (1) *the ratio between the unit stress and the corresponding unit strain is constant,* and (2) *upon removal of the external forces which produced the strain, the piece will regain its original form.* Since the elastic limit is a unit stress it is usually expressed in pounds per square inch. Its value depends chiefly on the nature of the material and the character of the stress, but it is also affected by other influences which for present purposes need not be considered.

Experiments have shown further that after the *elastic limit* has been reached, comparatively large increments of strain produce comparatively small increments of stress, and that upon the subsequent removal of the external forces the piece is found to be permanently deformed. Thus under a tensile unit stress a few thousand pounds above the elastic limit a steel bar 30 ft. long will suffer a permanent elongation of several inches. That circumstance alone serves to show that the unit stresses in a structure should never be allowed to reach the elastic limit. But even if the resulting permanent deformation should be unobjectionable, which is rarely or never true, experiments have also shown that the frequent development of unit stresses somewhat beyond the elastic limit may eventually produce complete rupture in the piece.

The elastic limit of structural steel is about 55 to 60 per cent of the ultimate strength.

7. Modulus of Elasticity. The constant ratio (below the elastic limit) between the unit stress, S/a, and the corresponding unit strain, Δ/l, is called the *modulus (measure) of elasticity*. Its value depends on the nature of the material and the character of the stress. If E denotes the modulus of elasticity,

$$E=\frac{Sl}{a\Delta}. \qquad (1)$$

Since the unit strain, Δ/l, is an abstract quantity, E is expressed in the same terms as the unit stress S/a—i.e., usually in pounds per square inch. If the strain is equal to the original length of the piece, $\Delta=l$, and if this strain is produced without exceeding the elastic limit, $E=S/a$. The modulus of elasticity may therefore also be defined as the unit stress when the length of a bar has been doubled, provided the elastic limit is not exceeded. In reality, no known material meets this requirement except pure rubber, and that only to an approximate degree. The value of the elastic limit has therefore to be computed by Eq. (1) from data obtained by measurements within the elastic limit.

For the common materials of construction, the modulus of elasticity has approximately the same value in tension and compression. The value of E for rolled or forged steel *of all grades* ranges from about 28,000,000 to 30,000,000. Assuming $E=30{,}000{,}000$, a unit strain of 0.001 will develop a unit stress of 30,000 lbs. per square inch. For a bar 25 ft. long a unit strain of 0.001 is equivalent to a total strain of 0.025 ft. or 0.3 in.

The modulus of elasticity is a direct criterion of the stiffness or rigidity of a material, i.e., the higher the modulus, the greater the stiffness.

The coefficient of expansion of steel for 1° F. is 0.0000065. This is therefore also the unit strain in a bar that will have to be developed by axial forces for every degree of change in temperature if the length of the bar is to remain constant. The corresponding unit stress is 0.0000065×30,000,000 = 195 lbs. per square inch. Hence if the ends of a bar are fixed, a change in temperature of 103° F. develops a unit stress of 195×103, i.e., about 20,000 lbs. per square inch, irrespective of the length or sectional area of the bar. Under falling temperatures this stress will be tensile and under rising temperatures, compressive.

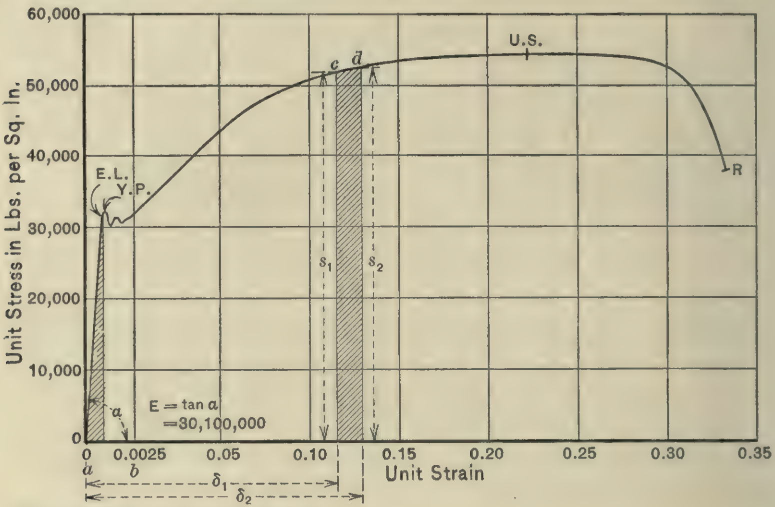
8. Yield Point. When a unit stress somewhat beyond the elastic limit is reached, the strain of certain materials increases for a time without an increase of load, or even under a slight decrease of load. The strain developed during that period is much greater than that within the elastic limit. The unit stress at which this action begins is called the *yield point* of the material. In commercial testing the yield point is easily recognized by the sudden drop of the scale beam of the testing-machine, and by other simple means, whereas the elastic limit can be found only by means of delicate measuring apparatus. In practice the term 'elastic limit' is often loosely used in the sense of 'yield point.' The yield point of structural steel is about 0.6 of the ultimate strength.

9. Stress-Strain Diagram. A complete record of the behavior of a piece tested to destruction under gradually increased loading is furnished by 'a stress-strain diagram.' Such a diagram for an actual tensile test of a piece of structural steel is shown in Fig. 1. The unit stresses are laid off as ordinates and the unit strains as abscissas from the origin of coordinates.

The observed values of the unit stresses at critical points are as follows: Elastic limit (E.L.), 31,900 lbs. per square inch; yield point (Y.P.), 32,300 lbs. per square inch; ultimate strength (U.S.), 54,000 lbs. per square inch; rupture (R.), 37,900 lbs. per square inch. The ratios of the elastic limit and the yield point to the ultimate strength are 59 and 60 per cent respectively. The observed values of the unit strains at critical points are as follows: At E.L., 0.00106; at Y.P., 0.00113; at U.S., 0.222; at R., 0.332.

Since the ratio of unit stress to unit strain is constant from the origin of coordinates to E.L. the line is straight between these points. That ratio, which is equal to the modulus of elasticity, E, of the material, is represented by the tangent of the angle α, between the inclined line and the axis of X. In this case $E = 31{,}900/0.00106 = 30{,}100{,}000$.

The unit strain at the elastic limit is so small that it had to be shown to an enlarged scale in the diagram, in order that the line from the origin of coordinates to E.L. might not appear to be coincident with the axis of *Y*.



The horizontal scale for the distance *ab* is ten times greater than for the remaining distance.

Fig. 1.

Until the ultimate strength, U.S., is reached the lateral strains, or deformations at right angles to the longitudinal axis of the piece, are, for steel, only about one-third as great, per unit of length, as the longitudinal strains. Beyond the ultimate strength the lateral strains, as well as the longitudinal strains, increase rapidly in the vicinity of some cross-section at which the piece develops a so-called 'neck.' The area of this cross-section decreases rapidly under decreasing loads until the piece ruptures at that section. The 'reduction of area' of cross-section at the neck, as measured after rupture, was in this case 67 per cent, and the 'percentage of elongation,' on an original 'gauge length' of 8 ins. across the neck was 33.

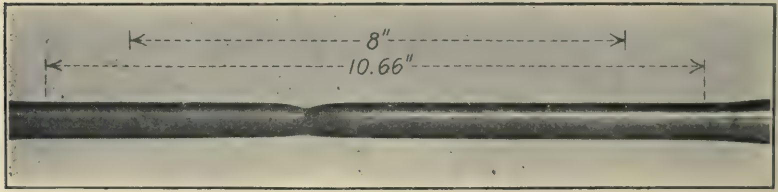


Fig. 2.

Fig. 2 is a reproduction of a photograph of a part of the specimen whose test record is shown in Fig. 1. The original diameter of the specimen was 0.635 in., and the final diameter at the neck, 0.365 in. The distance *ab*, which measured originally 8 ins., was elongated to 10.66 ins., according to measurements taken after rupture.

This record is fairly characteristic of some of the principal physical properties of structural steel.

10. Modulus of Resilience. Referring to Fig. 1, the unit stress at the elastic limit, as stated in the preceding article, is 31,900 lbs. per square inch, and the unit strain 0.00106. The work done by a force acting on a body is the product of the force and the distance through which the body has moved in the direction of the force. The force which produced the deformation in the present case increased uniformly from zero to 31,900 lbs. The work done on each cubic inch of the material, in developing the elastic limit, is therefore the product of the average force, 15,950 lbs., and the distance 0.00106 in., or 16.9 in.-lbs. This quantity is called the *modulus of resilience*, and it is represented graphically by the area of the shaded triangle in the figure. If s_e denotes the unit stress at the elastic limit, then s_e/E is the corresponding unit strain, and the general expression for the modulus of resilience is

$$Q=\frac{s_e^2}{2E}. \qquad (2)$$

The total resilience, within the elastic limit, of a bar n cubic inches in volume, is nQ, which therefore represents the work that must be done to develop the elastic limit throughout the bar. Within the elastic limit of the material a body may be likened to a perfect spring—that is to say, the work done in deforming the body is equal to the energy stored within the body.

The modulus of resilience is a direct measure of the shock-resisting property of a material; for the effective energy of a shock—that is to say, that part of the energy expended in causing deformation in the body against which the shock is directed—may be expressed in the same units, e.g., inch-pounds, as the total resilience of the body. In order that a material may be well adapted to resist shock without developing stresses beyond the elastic limit, it is seen from Eq. (2) that the modulus of elasticity, E, should be low, and the elastic limit, s_e, high. But a relatively low value of E implies a correspondingly high unit strain for a given unit stress, and therefore relatively great deflections in structures built of such material. In structures subjected to shocks, as bridges, for example, it is desirable that the material should possess a proper degree of resilience, but excessive deflections are on the other hand objectionable, as will be seen later. For that reason, and because Q varies as the second power of s_e and the first power of E, it is better that adequate resilience should be ensured by a relatively high elastic limit, rather than by a relatively low modulus of elasticity.

Referring again to Fig. 1, let s_1 and s_2 denote unit stresses so nearly equal that the line *cd* in the stress-strain diagram may be regarded as a straight line. If δ_1 and δ_2 denote the corresponding unit strains, $\delta_2 - \delta_1$ equals the increment of unit strain corresponding to the increment $s_2 - s_1$, of unit stress. The work done on each cubic inch of material in increasing the unit stress from s_1 to s_2 is then the product of the average force, $\frac{1}{2}(s_1 + s_2)$, and the distance $\delta_2 - \delta_1$, i.e., it is equal to the area of the shaded trapezoid in the figure. Thus it is seen that the total work per cubic inch of material that must be done to produce rupture is represented by the total area included between the stress-strain curve and the axis of X.

11. The Working Stress for any piece in a structure is the unit stress for which that piece was 'proportioned'— that is to say, the unit stress according to which its dimensions were determined. It is therefore the unit stress in the piece when the structure to which the piece belongs is subjected to the loads for which it was designed, provided the conditions assumed in the computation of the stress in the piece accord strictly with the actual conditions of service. Since that is at best only approximately true, and since, moreover, adequate provision must be made against contingencies, the working stress should be well below the elastic limit of the material.

The ratio of the ultimate strength of the material to the working stress is sometimes called the 'safety factor.' This term is misleading, however, because, as previously explained, the elastic limit rather than the ultimate strength indicates the safe limit of stress.

12. The Design of a Structure such as a bridge or a roof consists essentially of the following procedure:

1. The fixing of the general dimensions of the structure.
2. The determination of the reactions at the supports for the known or assumed loads.
3. The computation of the stresses from the external forces, i.e., the loads and reactions.
4. The proportioning of every part of the structure according to the assumed working stresses.
5. The computation of the weight of the structure, and in some cases, the distribution of the weight over different parts of the structure.
6. The revision of the design if the computed weight, or its distribution, does not agree within sufficiently close limits with the conditions assumed in the computation of the stresses.

In structures of certain types, the reactions, or the stresses, or both, cannot be determined by the application of the principles of statics, their values being influenced more or less by the elastic deformation of the structure. The analysis of such structures is said to be 'statically indeterminate.'

The deformation of a structure under the action of the external forces produces a movement in space of every part of the structure, including even the points at which the structure is supported, since the supports themselves cannot be perfectly rigid. Moreover, since every part of the structure is deformed, if only by its own weight, relative movement takes place between every point in the structure with respect to every other point. If the points of support are assumed to be fixed, or to have the same vertical movement, the vertical component of the relative movement of any given point of the structure with respect to any point of support is called the vertical deflection, or simply the 'deflection,' of the given point.

The reactions at the supports, and therefore the stresses in the structure, are affected by the relative movements of the points of application of the loads, and the stresses are likewise affected by the angular movement between the different parts of the structure; but these effects are so small that for practical purposes they may be neglected in structures in which the stresses are otherwise 'statically determinate.' That is to say, for the analysis of stresses such structures are regarded as perfectly rigid.

Volume I of this treatise will be limited essentially to structures in which the stresses are statically determinate in the sense just stated. It will treat of methods of finding the reactions, the stresses and the deflections from loads to which structures of the types to be considered are commonly subjected under service conditions.

While some knowledge of mechanics on the part of the reader may be assumed, certain principles especially applicable to the analysis of stresses will be briefly reviewed for the sake of continuity and completeness of treatment, and on account of their fundamental importance.

CHAPTER II

THE EQUILIBRIUM OF COPLANAR FORCES

13. A Force is fully known when its *magnitude,* its *position* and its *direction* are known. The force may then be represented by a straight line and an arrow-head. The length of the line represents, to any desired scale, the magnitude of the force; the position of the line agrees with the position of the line of action of the force; and the arrow-head indicates the direction along this line in which the force is exerted. In statics a force is expressed in units of weight—usually in pounds, thousands of pounds, or tons.

In so far as its effect on the equilibrium of a body is concerned, a given force may be assumed to be applied to the body at any point in its line of action.

14. The Moment of a Force about a point is the product of the magnitude of the force by the perpendicular distance from its line of action to the point. The moment is a measure of the tendency of a force to produce rotation about the given point. The point is called the *moment center,* and its perpendicular distance from the line of action of the force is called the *lever arm* of the force with respect to the point. A moment is said to be *positive* when it tends to cause rotation about the moment center in the same direction as the hands of a clock, and *negative* when it tends to cause rotation in the opposite direction. Positive moments are distinguished by the sign + and negative moments by the sign −. The *resultant moment* of a number of forces in the same plane about a common moment center is equal to the algebraic sum of the moments. Moments are usually expressed in foot-pounds, thousands of foot-pounds, inch-pounds, or thousands of inch-pounds.

Two parallel forces of equal magnitude acting in opposite directions form a *couple.* The moment of a couple is the product of the magnitude of one of the forces by the normal distance between the lines of action of the forces. This moment is also the resultant moment of the two forces constituting the couple about any point in their plane. That is to say, the tendency of a couple to produce rotation is the same with respect to any point in its plane. Hence, in so far as the effect of a couple on the equilibrium of a body is concerned, a given couple acting in a given plane may be assumed to be applied to the body in any position in that plane; or it may be assumed to be replaced by any other couple having the same moment applied in any position in the given plane.

15. Resolution and Combination of Forces. *Resolution of a Force.* A force may be *resolved* into two equivalent forces lying in the same plane with the given force in the following manner: In Figs. 1 (*a*) and (*b*) let F be the force to be resolved into two equivalent forces in the directions OB and OC. Let OA represent the magnitude of F. Complete the parallelograms as shown. Then OB and OC represent the desired equivalent forces f_1 and f_2 to the same scale to which OA represents F. The forces f_1 and f_2 are called the *components* of F, and F is the *resultant* of f_1 and f_2. Note that in such a diagram all three arrow-heads must point either *toward* or *away* from O, their point of intersection. A force may thus be resolved into two components at any point on its line of action. The effect of the two components on the equilibrium of a body is the same as that of their resultant.

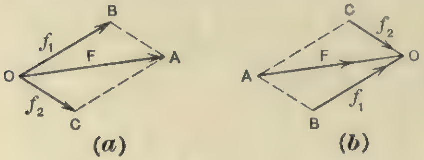


FIG. 1.

The truth of the following statements is either self-evident or nearly so:

1. A force cannot be resolved in two directions of which its own direction is one.
2. The lines of action of the two components cannot be coincident, unless they are coincident with the line of action of the original force.
3. The resultant moment of the two components about any point on their resultant is zero.
4. A force equal and opposite to the resultant, and having

the same line of action, will hold the resultant, and therefore its components, in equilibrium.

Combination of Forces. Two non-parallel forces may be *combined* into a single equivalent force—their resultant—by reversing the process just described. Thus in Fig. 2, r_1 is the resultant of forces 1 and 2, r_2 is the resultant of forces r_1 and 3, etc., and R is the resultant of forces 1 to 5, inclusive. The force E, equal and opposite to R, and having the same line of action, will hold R, and therefore forces 1 to 5, inclusive, in equilibrium. This force is called the *equilibrant* of forces 1 to 5, inclusive.

If forces 1 to 5, inclusive, are the only forces acting on a body, they will impart to the body a motion of translation in the direction of the resultant R; but if the force E is added to the system the body will be in equilibrium.

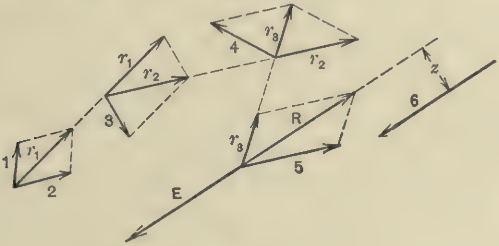


Fig. 2.

Suppose that the original force system had embraced another force, 6, Fig. 2, equal and parallel to R, but opposite in direction. Such a force cannot be combined with R, but forms a couple with R whose moment is Rz. If forces 1 to 6, inclusive, are the only forces acting on a body, they will impart to the body a motion of rotation in a clockwise direction. Equilibrium can then be established only by the application of another couple having an equal moment in the opposite direction. If forces 6 and R are parallel, but not equal in magnitude and opposite in direction, their resultant is a single force, as will be shown later (Art. 20).

Thus it is seen that if a system of forces lying in a common plane is not in equilibrium its resultant is either a single force (the usual case) or a couple (the exceptional case).

The Resultant of a Single Force and a Couple lying in a common plane, is a single force in the same plane. Thus in Fig. 3, let

F (shown by a full line) be the force and fz the couple. The latter may be replaced by the couple shown by broken lines, in which $z_1 = fz/F$. The two equal and opposite forces to the left may be removed without affecting the equilibrium of the body, so that finally $R(=F)$ is the resultant of the original force and couple. This resultant is equal and parallel to the original force, and acts in the same direction.

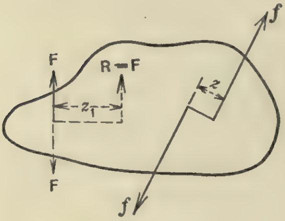


FIG. 3.

Conversely, let it be assumed now that $R(=F)$ is the only force acting on the body. Then at a distance z_1, perpendicular to the direction of R, assume two equal and opposite forces, F, to be applied. These forces will not affect the equilibrium of the body; but the three equal forces consist now of a couple whose moment is Fz_1 and the single force F, shown by a full line.

Hence, in general, *a force may be shifted any distance perpendicular to its direction without affecting the equilibrium of a body, provided a couple is added in the plane in which the force is shifted, whose moment is the product of the force by the distance through which it is shifted, and which tends to cause rotation in the same direction as the original force did about any point in its new position.* This principle is one of great importance in the analysis of stresses.

16. Conditions of Equilibrium of Coplanar Forces. Forces whose lines of action lie in a common plane are said to be *coplanar.* In order that a body may be in equilibrium under the action of a system of coplanar forces, the following three laws of equilibrium must be fulfilled:

I. *The algebraic sum of the vertical components of the forces must equal zero.*
II. *The algebraic sum of the horizontal components of the forces must equal zero.*
III. *The algebraic sum of the moments of the forces about any point in their plane must equal zero.*

The external and internal forces acting on or in a structure can usually be grouped into various systems of coplanar forces; so that the above three laws of static equilibrium suffice for their analysis. In fact the application of statics to structures is based essentially on these three fundamental laws, which will be referred to hereafter as $\Sigma V=0$, $\Sigma H=0$ and $\Sigma M=0$. Unless otherwise stated it is also to be understood hereafter that the forces under consideration are coplanar.

If Laws I and II are satisfied the forces acting on the body can cause no motion of translation; and if Law III is satisfied they can cause no motion of rotation. In considering the equilibrium of a system of forces it is usually convenient to regard the forces as resolved into vertical and horizontal components. Laws I and II may, however, be more broadly stated by saying that if the algebraic sum of the components in *any two directions* is zero, the forces acting on the body can cause no motion of translation.

The truth of the following statements is either self-evident, or apparent from the foregoing:

1. If a single force is applied to a body, the body cannot be in equilibrium.
2. If two forces are applied to a body, the body can be in equilibrium only if the forces are equal and opposite, and have the same line of action. Such pairs of forces may be applied at any points of a body without affecting its equilibrium.
3. If three non-parallel forces are applied to a body, the body can be in equilibrium with respect to motion of rotation, only if the three forces intersect at a common point. Otherwise the moment of each force about the intersection of the other two will be unbalanced. It follows, therefore, that if the position of two of the forces and a point on the line of action of the third force are known, the line of action of the latter may be found by connecting the known point with the point of intersection of the two forces whose positions are known.
4. If any number of forces are applied to a body, and if that body is then in equilibrium, the resultant of any group of these forces will balance the resultant of the remaining group. If the two resultants are single forces (the usual case) the resultant of either group is the equilibrant of the other group. If the two resultants are couples (the exceptional case) the

resultant of either group will hold the forces in the other group in equilibrium. Finally, any single force is the equilibrant of all the other forces.

5. Assuming the resultant of a given system of forces to be a single force, the moment of that resultant about any point on its line of action is zero; hence the resultant moment of the forces about any point on their resultant is zero.

17. Laws of Equilibrium in Terms of Moments. It has been seen that a system of coplanar forces is in equilibrium if $\Sigma V=0$, $\Sigma H=0$, and $\Sigma M=0$. It is frequently convenient, however, to express the laws of equilibrium in terms of moments only.

(*a*) *Parallel Forces. A system of parallel forces is in equilibrium if the resultant moments of the forces about any two points not lying in a line parallel to the forces are zero.* This may be proved as follows:

If the direction of the parallel forces is not vertical the entire system may be revolved till the forces are vertical, as in Fig. 4, without affecting the conditions of equilibrium. When the forces are vertical, $\Sigma H=0$. The system is therefore in equilibrium if $\Sigma V=0$ and $\Sigma M=0$.

Let M_A denote the resultant moment of the forces 1–7 about any point A. If $M_A=0$, the resultant of the forces 1–7 cannot be a couple; the resultant must then either be zero, or a vertical force (ΣV) passing through A. The resultant moment of the forces 1–7 about any second point, B, not lying in a vertical through A, but at a horizontal distance a from A, is $M_B=\Sigma V\times a$. Therefore the moment M_B about the second point cannot be zero unless $\Sigma V=0$. Hence if M_A and M_B are both zero, the forces 1–7 are in equilibrium.

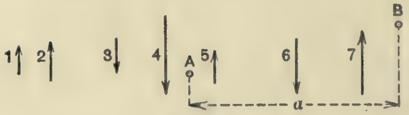


FIG. 4.

(*b*) *Non-Parallel Forces. A system of non-parallel forces is in equilibrium if the resultant moments of the forces about any three points not lying in the same straight line equal zero.* This may be proved as follows:

Let the forces 1–5, Fig. 5, be assumed to have any inclinations. Let the resultant moments M_A, M_B and M_C, of the forces 1–5 about any points A, B and C not lying in the same straight line, equal zero. If the resultant moment, M_A, of a system of forces about any single point A is zero, the resultant of the forces cannot be a couple. The resultant is then either zero or a force passing through A. Similarly, if $M_B=0$, the resultant is either zero or a force passing through B. Hence if M_A and M_B are both zero, the resultant of the forces 1–5 is either zero or a force whose line of action lies in the direction AB. If, further, M_C is zero, the resultant must be zero and the forces are in equilibrium; for since A, B and C are not in a straight line the resultant cannot pass through all three of these points. Hence, if $M_A=0$, $M_B=0$, and $M_C=0$, the forces must be in equilibrium.

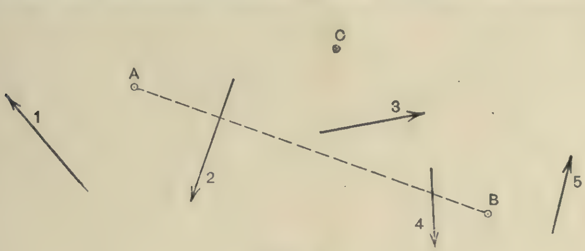


FIG. 5.

18. Concurrent Forces; Equilibrant by Force Polygon. Forces are said to be *concurrent* or *non-concurrent* according to whether their lines of action do, or do not, meet at a common point.

A system of six concurrent forces, 1–6, is shown in Fig. 6 (*a*). Let each force be resolved at O into horizontal and vertica- components. Let V_N, V_S, H_E, and H_W, Fig. (*b*), denote the arithmetic sums of the components whose directions are northward, southward, eastward and westward, respectively. If the six forces are in equilibrium they must fulfill the three laws of equilibrium (Art. 6). Evidently $\Sigma V=0$ if $V_N=V_S$, and $\Sigma H=0$ if $H_E=H_W$. But if these two conditions are satisfied, the third ($\Sigma M=0$) is necessarily met in the special case of concurrent forces; for the moment of H_E about any point is then equal and opposite to that of H_W, and similarly the moments of V_N and V_S about any point are equal and opposite. Or again, if $\Sigma V=0$ and $\Sigma H=0$, the resultant of the forces (which for concurrent forces must be either a single force through O, or zero, but cannot be a couple) is zero; whence $\Sigma M=0$.

Thus it is seen that *for concurrent forces there are only two independent equations which jointly serve to express the conditions under which the system is in equilibrium.* These equations may be utilized to determine two—but not more than two—unknown elements, the magnitude (including direction) and the position (including direction) of a force being each regarded as one of these elements. The application of these equations to the determination of the equilibrant will be now considered.

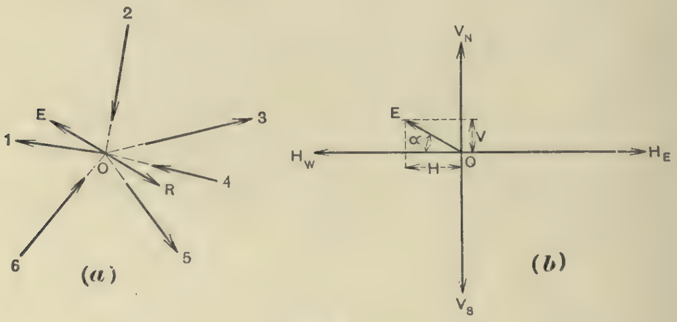


FIG. 6.

Let it be assumed that the forces, 1–6, Fig. 6 (*a*), are not in equilibrium, and let it be required to find the magnitude and position (including direction) of their equilibrant. In the general case neither ΣV nor ΣH is equal to zero. If $V_S>V_N$, the vertical component $V(=V_S-V_N)$ of the equilibrant, E, must act northward from O as shown in Fig. (*b*). Similarly, if $H_E>H_W$, the horizontal component, $H(=H_E-H_W)$, must act westward from O.

Graphically, E may be determined in magnitude and direction as the diagonal of the rectangle whose sides are V and H. This force E applied at O, Fig. (*a*), will hold forces 1–6 in equilibrium. The force R, equal and opposite to E, is the resultant of forces 1–6.

The values of V and H may obviously be computed, if desired, from the known magnitudes and directions of the forces 1–6. Similarly, by computation, $E=(H^2+V^2)^{\frac{1}{2}}$, and $\tan\alpha=V/H$.

The most convenient method of finding E graphically is as follows:

Beginning at a point *a*, Fig. 7 (*b*), lay off successively the forces 1, 2, 3, etc., parallel to the corresponding forces in Fig. (*a*), and in the directions indicated by the arrow-heads of the given forces, the length of each line representing to a chosen scale the magnitude of the force. The closing line *ga* of the polygon—called a *force polygon*—represents to scale the magnitude of the equilibrant, and in direction from *g* to *a* the direction of the equilibrant. Since the arrows point in the same direction around the polygon, the latter may be regarded as the path of a moving point starting from *a*, leading successively to *b*, *c*, *d*, etc., and finally returning to the starting point *a*. Obviously, therefore, the point must have moved as far north as south ($\Sigma V = 0$), and as far east as west ($\Sigma H = 0$). It is also evident that the order in which the elements of the path (the forces) follow each other is a matter of indifference, provided the directions indicated by the arrows in Fig. (*a*) follow continuously in the same direction around the polygonal path described. Thus the path 6–5 would lead from *e*, via *f′*, to the same point *g*, as the path 5–6, via *f*. Or, in general, if a point in moving from *a* to any point *z* by any two paths moves the same distances north, south, east and west, respectively, by both paths, the relative positions of *a* and *z* will be the same.

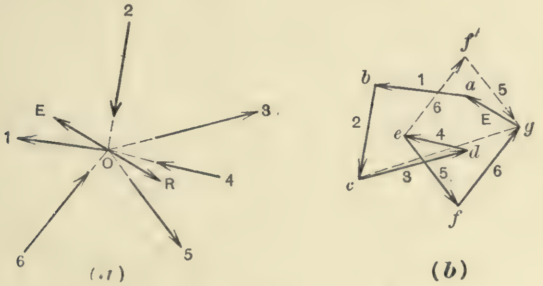


FIG. 7.

A system of forces is balanced as regards translation when the algebraic sums of the vertical and horizontal components of the forces equal zero, in the same sense in which a survey of the boundary lines of an area is balanced when the algebraic sums of the latitudes and departures equal zero.

From the foregoing it also follows that a line drawn from any apex of the force polygon as, for example, c, to any other apex, g, represents the equilibrant of all the forces to one side, and the resultant of all the forces to the other side. Thus if the arrow-head is pointed from c to g, $c\ g$ is the equilibrant of forces 1, 2 and E, and the resultant of the remaining forces, 3–6, or vice-versa if the arrow-head is pointed in the opposite direction.

Again, if the forces follow in the same direction around the force polygon, any single force is the equilibrant of all the other forces. Conversely, a force equal and opposite to any single force is the resultant of all the other forces.

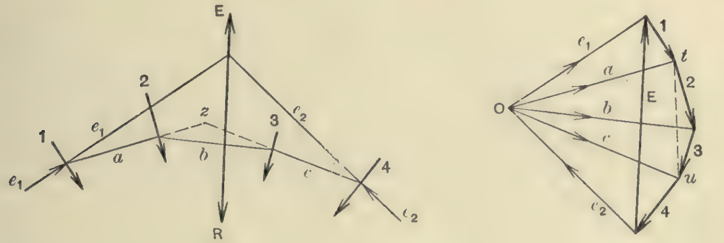
19. Non-Concurrent Forces; Equilibrant by Equilibrium Polygon. The *magnitude* and *direction* of the equilibrant of a system of non-concurrent forces depend only on the fulfillment of the first two laws of equilibrium, which relate to equilibrium as regards translation, i.e., $\Sigma V=0$ and $\Sigma H=0$. They may therefore be found by means of a force polygon in precisely the same manner as for concurrent forces. The *position* of the equilibrant must be such as to satisfy the third law of equilibrium, which relates to equilibrium as regards rotation, i.e., $\Sigma M=0$. If the angles of intersection of the forces are favorable the equilibrant may be completely determined by the method illustrated in Fig. 2, Art. 15. This method is not applicable, however, to the common case of parallel forces, nor is it practicable if the forces are nearly parallel. A general method by which the equilibrant may be fully determined graphically will now be considered.

Let it be required to find the equilibrant of forces 1–4, Fig. 8 (a). The magnitude and direction of the equilibrant, E, is found by drawing the force polygon shown by heavy lines in Fig. (b). The position of E in the *space diagram*, Fig. (a), remains to be found.

Select a point O in such a way that lines e_1, a, b, etc., drawn from that point to the extremities of the forces, will make no sharp angles with their lines of action. The point O is called the *pole*, and the lines radiating from the pole are called the *rays* to the force polygon. Rays e_1 and e_2 are components of the equilibrant E. With the arrows pointing in the direction shown, ray a is the resultant of e_1 and 1, b is the resultant of a and 2, c is the resultant of b and 3, and e_2 is the *equilibrant* of c and 4, and therefore the equilibrant of e_1, 1, 2, 3 and 4.

Let it be required, first, to assign such positions to e_1 and

e_2 in the space diagram, Fig. (*a*), that they will jointly hold the given forces 1–4 in equilibrium. From any point on 1 draw e_1 and *a* parallel to the corresponding rays to the force polygon; produce *a* to its intersection with 2, draw *b* parallel to ray *b*, and continuing, similarly draw *c* and e_2. The resulting figure forms part of an *equilibrium polygon*, which may be completed by prolonging e_1 and e_2 to their intersection. The sides of this equilibrium polygon are called its *segments*. Segment *a* denotes the position of the resultant of e_1 and 1, whose magnitude and direction are shown by ray *a*. Similarly segment *b* indicates the position of the resultant of *a* and 2, whose magnitude and direction are shown by ray *b*. Finally, e_2 denotes the position of the resultant of *c* and 4, that is to say, the resultant of e_1 and the given forces 1–4. (The direction of this resultant is opposite to that of the arrow-head shown). If, therefore, a force whose magnitude and direction are shown by ray e_2 is placed in the position denoted by segment e_2, the entire force system, consisting of the given forces 1–4 and the auxiliary forces e_1 and e_2, will be in equilibrium. The forces e_1 and e_2 may now be replaced by their resultant *E* drawn through their intersection, and parallel to *E* in the force polygon. The equilibrium polygon is thus closed, and *E* is the required equilibrant of the given forces. The force *R*, equal and opposite to *E* and in the same position, is the resultant of the given forces. Summarizing: *to ensure equilibrium of translation, the force polygon must close, and to ensure equilibrium of rotation, the equilibrium polygon must close.*



(*a*) (*b*)

FIG. 8.

Considering the auxiliary forces e_1 and e_2 removed from the space diagram, it will be seen that in Fig. (*b*) the lines parallel to any force and the adjacent segments of the equilibrium polygon form a triangle. The equilibrium polygon may therefore be regarded as a frame, hinged at its apexes, in which *the stresses in any two adjacent pieces hold in equilibrium the external force acting at their intersection.* The magnitudes and directions of these stresses are shown by the corresponding rays to the force polygon.

It is instructive to observe the following relations between Figs. (*a*) and (*b*):

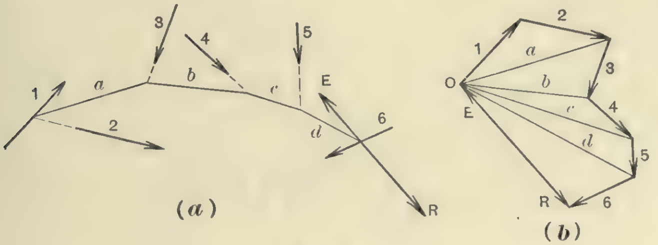
1. Any two consecutive forces, as, for example, 2 and 3, are *separated* (so to speak) by the intervening ray, *b*, in the force polygon, and *united* by the corresponding segment, *b*, in the equilibrium polygon.
2. Since the pole may be chosen at random, the direction of one, or two, segments in the equilibrium polygon may be predetermined, if desired. Thus if the direction of segment e_1 is to be horizontal, O may be chosen at any point on a ray, e_1, drawn horizontally through the intersection of E and 1 in the force polygon. If it is desired further that segment e_2, for example, shall slope upward to the left at an angle of 45° with the horizontal, O may be located at the intersection of ray e_2, drawn in that direction, with the horizontal ray e_1.
3. Since the location of the first segment drawn in the equilibrium polygon may be chosen at random, it may be located so as to traverse any desired point.

From the foregoing, it may be seen by strict analogy that the equilibrant of any number of consecutive forces, as, for example, 2 and 3, may be determined by producing the segments *a* and *c* to either side of these forces, to their intersection *z*, and by drawing their equilibrant, whose magnitude is *tu*, Fig. (*b*), through *z*, and parallel to *tu*.

Since any segment of the equilibrium polygon denotes the position of the resultant of all the forces to the left of that segment, including e_1, it must denote also the position of the resultant of all the forces to the right, including e_2. For the entire system of forces, e_1, 1, 2, 3, 4 and e_2, is in equilibrium, and the resultant of any group of these forces must balance the resultant of the remaining group.

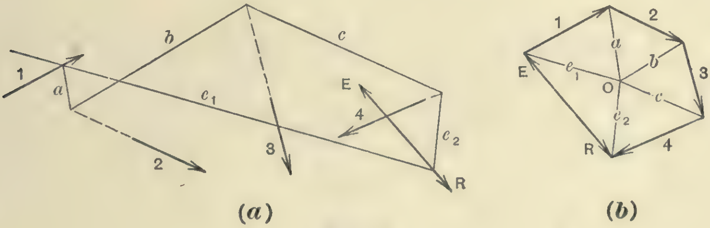
Pole at Apex of Force Polygon. If the inclinations of the given forces are sufficiently varied to ensure favorable angles of intersection, the construction of the equilibrium polygon may be simplified by selecting an apex of the force polygon as the pole.

Let 1–6, Fig. 9 (*a*), be the given forces. Draw the force polygon, Fig. (*b*), in which *E* is the equilibrant, and *R* the resultant. Select apex *O* as the pole, and draw rays *a*, *b*, *c*, and *d*. From the intersection of 1 and 2 in the space diagram, draw segment *a* of the equilibrium polygon parallel to ray *a*; from its intersection with 3, draw *b* parallel to ray *b*. Similarly, draw segments *c* and *d* parallel to rays *c* and *d*. Through the intersection of *d* with 6 draw *E*, the required equilibrant, parallel to *E* in the force polygon. Segment *a* evidently denotes the position of the resultant of forces 1 and 2; segment *b*, of forces 1, 2, and 3; and, in general, any segment denotes the position of the resultant of all the forces to the left of that segment. An equilibrium polygon of this character may therefore be called a *resultant polygon* (although it is in reality not a closed figure), to distinguish it from an equilibrium polygon of the type shown in Figs. 8 and 10, in which auxiliary forces, e_2 and e_1, are temporarily introduced.



(*a*) (*b*)

FIG. 9.



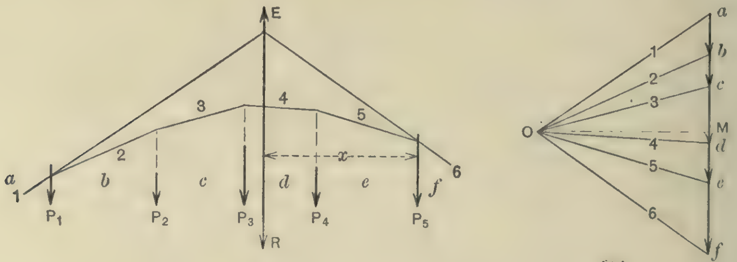
(*a*) (*b*)

FIG. 10.

Observe that in Fig. 2, Art. 15, such a resultant polygon was constructed with the aid of separate force polygons instead of a composite force polygon as in Fig. 9.

Pole at Interior of Force Polygon. Referring to Fig. 10 (*a*), the equilibrant of forces 1–4 will be determined by assuming the pole *O* at some central point in the interior of the force polygon, Fig. (*b*). Begin the construction of the equilibrium polygon, Fig. (*a*), at any point on 1, drawing e_1 and *a* parallel to the corresponding rays, Fig. (*b*). Continuing, draw *b*, *c* and e_2. The intersection of e_1 and e_2 is a point on the required equilibrant.

20. Equilibrant of Parallel Forces. The case of parallel forces is the most common one in the analysis of structures. For convenience the forces will be assumed to be vertical. Let it be required to determine the equilibrant of the five forces P_1–P_5, Fig. 11 (*a*), which are assumed to act in the same direction, since that is the usual condition in practice.



(*a*) (*b*)

FIG. 11.

The notation to be used here, and frequently hereafter, is that devised by Bow, and generally used in the graphic analysis of forces. By this system each force in the space diagram is distinguished by the letters in the two spaces on opposite sides of the force. Thus *ab* denotes the force P_1, *de* the force P_4, etc. In the force polygon the same letters appear at the extremities of the forces in question.

Construct the force polygon by laying off the forces *ab*, *bc*, *cd*, etc., to *f*. Then *fa*, in the direction from *f* to *a*, is the equilibrant in magnitude and direction, the force polygon being in this case reduced to a straight line, called a *load line*. Select a convenient point, *O*, as the pole and draw the rays 1, 2, 3, etc. To construct the equilibrium polygon: from any point on the force P_1 draw 1 and 2 parallel to rays 1 and 2, and continue the diagram in the usual way. The position of the equilibrant, E, is determined by the intersection of 1 and 6, as explained in the preceding article.

The horizontal distance, OM, from the pole to the force polygon is called the *pole distance*, although it is really a force. Its usefulness in the solution of certain problems will be seen later (Art. 37).

The equilibrant may be found by computation as equal in magnitude, and opposite in sign, to the algebraic sum of the given forces. Its horizontal distance from any point, as, for example, a point on the line of action of force P_5, is $x = M/\Sigma P$, in which M is the resultant moment of all the given forces about the assumed point, and ΣP the algebraic sum of these forces.

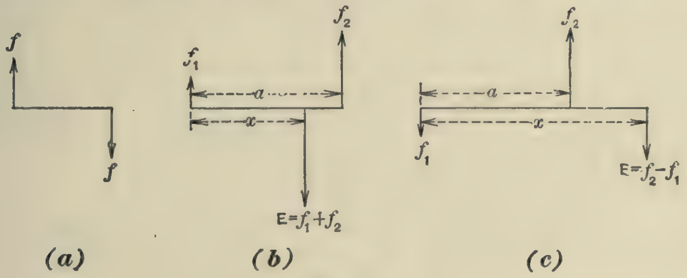


Fig. 12.

If the algebraic sum of the given forces is zero it indicates that they are either in equilibrium or that their resultant is a couple. In the latter case their equilibrant is an equal couple acting in the opposite direction.

Two Parallel Forces. If the two parallel forces, f, Fig. 12 (a), are *equal* and act in *opposite* directions, they constitute a couple, and their equilibrant (not shown) is then an equal and opposite couple.

If the two parallel forces, f_1 and f_2, Fig. (b), are *unequal* and act in the *same* direction, the equilibrant is equal to their sum. Its distance x from either force may be found by moments about any point in that force. Thus $x = a \times f_2/(f_1 + f_2)$. Since $x < a$ the equilibrant must lie *between* the given forces, and it must act in the opposite direction.

If the two parallel forces, f_1 and f_2, Fig. (c), are *unequal* and act in *opposite* directions, the equilibrant is equal to their algebraic sum. Its distance from the smaller force, f_1, is $x = a \times f_2/(f_2 - f_1)$. Since $x > a$ the equilibrant must lie *outside* of the given forces, on the side of the larger force, and it must act in the same direction as the smaller force.

Evidently any force in Figs. (*b*) and (*c*) is the equilibrant of the other two forces. A force equal and directly opposite to any force is the resultant of the other two forces; and conversely, the latter are the components of this resultant.

CHAPTER III

APPLICATION OF THE LAWS OF EQUILIBRIUM TO THE DETERMINATION OF REACTIONS

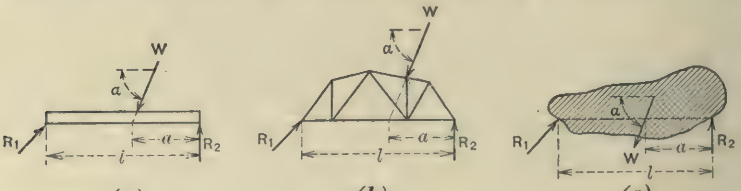
21. General Considerations. Before the stresses in a given structure from known or assumed loads can be determined it is necessary to find the reactions at the supports. In order that the structure as a whole may be in equilibrium the loads and reactions jointly must fulfill the three laws of equilibrium. Since the reactions must pass through the supports, one point on the line of action of each reaction is known. It remains then to determine the magnitudes, inclinations (instead of positions) and directions of the reactions.

A support is said to be *fixed* if it does not admit of motion of translation in any direction, although admitting freely of angular movement or rotation of the structure in a vertical plane under the deflections produced by the elastic deformation of the various parts subjected to stress from the external forces.

A *sliding*, *rolling* or *rocking* support is one which admits of motion of rotation, and also of motion of translation in a horizontal direction under friction which is assumed to be negligible. The reaction at a support of this kind is therefore necessarily vertical, whereas the reaction at a fixed support may have any direction consistent with the requirements of equilibrium for the structure as a whole. For a rolling or rocking support the error involved in the neglect of friction is unimportant, whereas for a sliding support it is relatively large.

The character of a structure does not affect the reactions from a given load system, provided (1) the reactions are statically determinate, (2) the points of support occupy the same relative positions towards the given loads, and (3) the conditions at the supports, in so far as these may influence the inclinations of the reactions, are the same. Thus in Fig. 1 (*a*) let R_1 and R_2 be the reactions from the load W acting on a beam. The horizontal distance, l, between the supports is called the *span*. Assume the right end of the beam to be supported on rollers so that, neglecting rolling friction, R_2 must act vertically; and assume the left end to be fixed in so far as translation (not rotation) is concerned. The inclination of R_1 may be found by recognizing that its line of action must pass through the intersection of W and R_2 (Art. 16), whose inclinations are known. The magnitudes of R_1 and R_2 may be found by drawing a force polygon of these forces and W, or by other means to be considered later.

If a, l, W and α have the same values in Figs. (*b*) and (*c*) as in Fig. (*a*), and if the conditions at the supports are as above described, the reactions R_1 and R_2 may be found as explained, and their values will be the same as in Fig. (*a*). It is to be understood, of course, that each structure is sufficiently strong to develop the required reactions, and sufficiently rigid to do so without distorting to a degree which would sensibly affect the values of a and l, or rather that of the ratio a/l.



(*a*) (*b*) (*c*)

FIG. 1.

Since the character of the structure does not affect the reactions, the structure will not be shown in connection with problems involving only the determination of the reactions from given loads.

22. Single Support. If a structure rests on a single support, as, for example, a locomotive turn-table, it is necessary for equilibrium (1) that the line of action of the equilibrant of the loads shall pass through the support, and (2) that the conditions at the support shall be such as to render possible the development of the equilibrant—that is to say, a reaction of the required magnitude, inclination and direction.

In Fig. 2, let P_1–P_4 denote the loads, and A the support. The reaction R, i.e., the equilibrant of the loads, may be found graphically by the method in Art. 19. For equilibrium it must pass through A. The forces V and H are its vertical and horizontal components, respectively. If the structure rests on rollers at A, H cannot be developed, and the structure cannot be in equilibrium unless R is vertical.

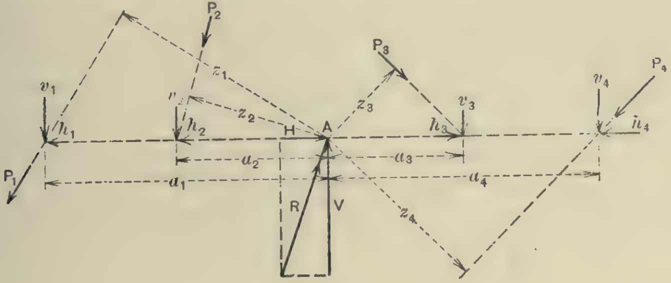


FIG. 2.

The reaction may also be found by computing the vertical and horizontal components of each load from its known magnitude, inclination and direction. Then $V=v_1+v_2+v_3+v_4$, and $H=h_1+h_2+h_4-h_3$. To investigate equilibrium of rotation, the lever arms, z_1–z_4 of the forces, or the lever arms a_1–a_4 of their vertical components may be calculated, or scaled from a large-scale drawing. Then for equilibrium of rotation: $P_1z_1+P_2z_2-P_3z_3-P_4z_4=0$, or $v_1a_1+v_2a_2-v_3a_3-v_4a_4=0$.

23. Two Supports, Both Fixed. In Fig. 3 let it be assumed that the supports A and B are both fixed, and let P be the resultant of the loads. Produce P to its intersection with AB, and resolve it into its vertical and horizontal components V and H. From $\Sigma M=0$, taking moments about B, $R_v=Va/l$, and from $\Sigma V=0$, $R_v'=V-R_v$. The value of R_v' might have been found from $\Sigma M=0$ by taking moments about A, whence $R'_v=V(l-a)/l$. From $\Sigma H=0$, $R_h+R_h'=H$, which is the only remaining independent equation of equilibrium, so that the values of R_h and R_h', the horizontal components of the reactions, are indeterminate, both as to magnitude and direction.

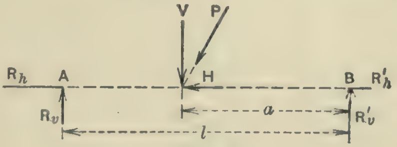


FIG. 3.

Thus it is seen that the reactions of a structure resting on two supports are statically indeterminate if both supports are fixed. It is usually assumed under these circumstances that the horizontal components of the reactions are equal, i.e., $R_h = R_h' = \frac{1}{2}H$. It is sometimes assumed that the lines of action of the reactions are both parallel to the resultant of the loads, i.e., $R_h = R_v\ H/V$ and $R_h' = R_v' H/V$.

If the resultant of the loads is vertical, as shown in Fig. 4, R_v and R_v' have the same algebraic values as in Fig. 3, but the reactions are nevertheless statically indeterminate if both ends are fixed. If one end is on rollers, the distance, l, between supports will change on account of the elastic deformation of the parts of the structure under stress. If both supports are fixed, however, this movement is prevented by the development of horizontal reactions, H, at the supports which, from $\Sigma H = 0$, must be equal and opposite, but whose magnitude and directions are statically indeterminate.

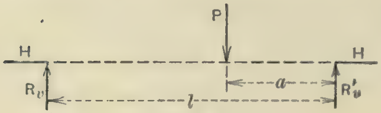


FIG. 4.

Thus it is seen that, in general, *the reactions of a structure resting on two supports are statically determinate only if one of the supports admits of translation, and both supports admit of rotation.*

24. Two Supports, One Fixed. (*a*) *Inclined Loads.* In Fig. 5, assume the left support fixed, and the right support on rollers. Let P be the resultant of the loads, which may be found graphically as in Art. 19, and let V and H be its vertical and horizontal components. Since R_1 is vertical, and since for equilibrium the three forces P, R and R_1 must intersect at a common point, AC is the line of action of R. From $\Sigma H = 0$, $R_h = H$, and R may be found by completing the force triangle for the forces meeting at A, or R and R_1 may be found directly by resolving P in the directions CA and CB.

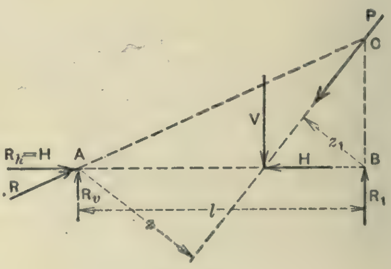
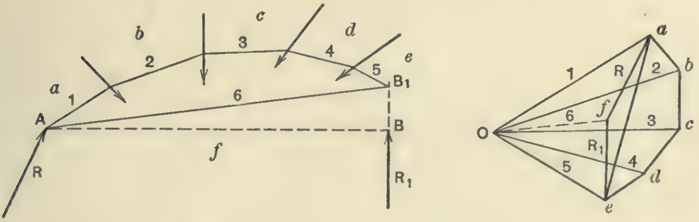


Fig. 5.

The reactions may be obtained by computation, if preferred, as in Art. 23, or from moments of P about A and B, i.e., $R_v=Pz_1/l$ and $R_1=Pz/l$. Finally, $R=(R_v^2+R_h^2)^{\frac{1}{2}}$. If the resultant, P, of the loads, is not known, this procedure may be applied to the individual loads (not shown), as in Fig. 2, Art. 22. If M and M_1 denote the resultant moments of these loads about A and B, respectively, $R_v=M_1/l$, and $R_1=M/l$. The algebraic sum of the horizontal components of the loads then equals $R_h(=H)$.

Graphically, the reactions may be found directly from the loads, instead of from their resultant, as follows: Let the four forces *ab*, *bc*, etc., Fig. 6, represent the loads. Draw the force polygon, Fig. (*b*), in which *ea*, in the direction from *e* to *a*, is the equilibrant. The required reactions must be components of this equilibrant. Beginning at A, the support at which the direction of the reaction is unknown, construct the equilibrium polygon, 1, 2, 3, etc., and from the intersection, B_1, of segment 5 with a vertical through B, draw the closing segment, 6. From the pole, O, draw ray 6, parallel to the corresponding segment of the equilibrium polygon, to its intersection, *f*, with a vertical rhrough *e*, and draw *fa*. Then $\overline{ef}=R_1$ and $\overline{fa}=R$, in magnitude and direction. Since the equilibrium and force polygons are in correct relations to each other (Art. 19) and are closed, the requirements for equilibrium of rotation and translation are both fulfilled. The forces R_1 and R are seen to be components (the former vertical) of the equilibrant *ea*, as required. The intersection of R and R_1, Fig. (*a*), will be a point on the resultant of the loads (corresponding to P in Fig. 5) whose line of action is parallel to *ae*, Fig. (*b*), and which is equal in magnitude and opposite in direction to the equilibrant *ea*.



(*a*) (*b*)

FIG. 6.

(*b*) *Vertical Loads.* If the loads are all vertical the reactions may be most readily found by computation. Thus in Fig. 7, if ΣP denotes the resultant of the loads, and a_1, a_2, etc., the distances from the loads P_1, P_2, etc., respectively, to the right support,

$$R = \frac{P_1a_1+P_2a_2+P_3a_3+P_4a_4}{l} = \frac{\Sigma P\times b}{l}, \quad . \quad . \quad . \quad (1)$$

and

$$R_1 = \Sigma P - R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

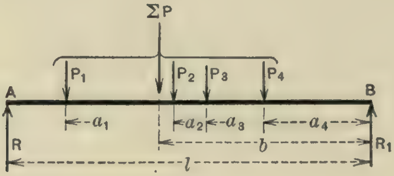


FIG. 7.

The reaction R_1 may be computed independently by taking moments about A, in which case the fulfillment of the condition that $R+R_1=\Sigma P$ serves as a check on the correctness of the results.

The reactions may be found graphically, as before, by means of an equilibrium polygon, Fig. 8 (a), which may be begun at any point, A_1 or B_1, on a vertical through A or B. In Fig. (b) the ray 6 is drawn parallel to the closing segment, 6, of the equilibrium polygon, and the components ef and fa of the equilibrant ea, Fig. (b), are then the required reactions R_1 and R, at B and A respectively.

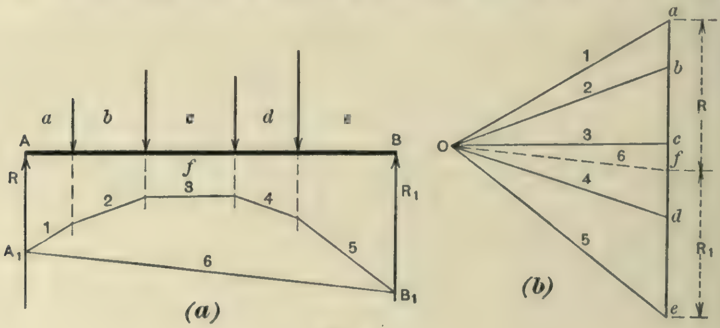


FIG. 8.

25. More Than Two Supports. If the number of supports exceeds two, the structure is said to be *continuous* over the number of supports in question, and the reactions are statically indeterminate. The three laws of equilibrium admit of the establishment of only three independent equations, from which at most three unknowns may be determined. Assuming the simplest conditions for a structure on three supports—namely, that the loads are all vertical and the supports are all on rollers—there will be only three vertical reactions. But since the external forces are all vertical it is implied that $\Sigma H=0$. There remain then only two independent equations, from $\Sigma V=0$ and $\Sigma M=0$, for finding three unknowns, and *the reactions are therefore statically indeterminate.*

It is a common error of beginners to assume that the three reactions may be derived from three equations expressing the conditions that the resultant moment of the external forces about each of the three points of support equals zero. As shown, however, in Art. 17, if the resultant moments of a system of parallel forces about two points (not lying in a line parallel to the forces) equal zero, the system is in equilibrium. It follows then as a corollary—not as an independent condition of equilibrium—that the resultant moment about any other point whatever must equal zero. This is equivalent to saying that if the resultant moments about each of two points of support are equated to zero, a third equation expressing this condition with respect to the third point of support, may be derived directly from the first two equations. The third equation is therefore identical with, and not independent of, the first two equations.

26. A Three-Hinged Arch, Fig. 9, is a structure which rests on two hinged supports, A and B (usually on the same level), whose distance apart is fixed, and which is also hinged at some intermediate point, C, usually a point of symmetry of the structure. The hinges are assumed to be frictionless—a condition which can be realized at best only approximately. Since the supports A and B are fixed, in the sense defined in Art. 21, the reactions would be statically indeterminate (Art. 23) except

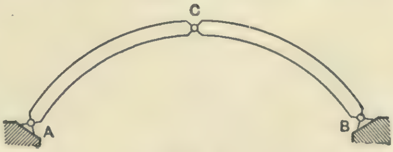


FIG. 9.

for the hinge C, whose existence supplies a fourth condition of equilibrium, namely, that the resultant moment of the external forces to *either* side of C about that point is zero. This implies that the resultants of these two sets of forces must pass through C, and that they must be equal in magnitude, but opposite in direction. The reactions for different conditions of loading may be readily found either graphically or by computation. In general, the reactions are both inclined, so that there are four unknowns, namely, the vertical and horizontal components of both reactions. If the relative positions of the three hinges are fixed, the reactions from a given load system are not affected by the character of the structure in other respects, so that the structure will hereafter be represented merely by a line which may have any curvature. In fact the simplest example of a three-hinged arch is that of two straight pieces, the one from A to C and the other from C to B.

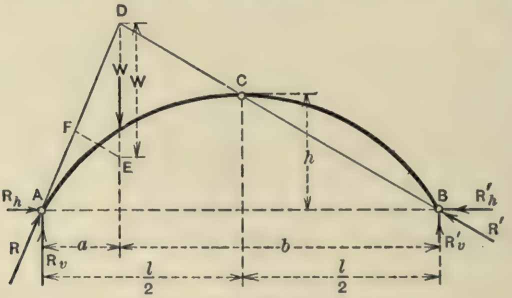


FIG. 10.

(*a*) *Single Vertical Load. Graphic method.* Referring to Fig. 10, the reaction R' at the foot of the unloaded segment must pass through C; for since this is the only force to the right of C, its moment about C must be zero. Produce BC to its intersection D with W. Then DA is the line of action of the other reaction R. From D lay off $\overline{DE}=W$, and draw EF parallel to BD. Then $R=\overline{FD}$ and $R'=\overline{EF}$.

Algebraic method. From $\Sigma M=0$, taking moments about A and B respectively,

$$R_v'=W\frac{a}{l}, \quad \text{and} \quad R_v=W\frac{b}{l}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From $\Sigma H=0$, $R_h=-R_h'$, and from the known slope of R', or by moments of R_h' and R_v' about C,

$$R_h=-R_h'=R_v'\frac{l}{2h}=W\frac{a}{2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The opposite signs for R_h and R_h' merely indicate that these forces act in opposite directions.

The vertical components of the reactions are seen to have the same values as for a beam of span l similarly loaded. This is true only for the usual case of A and B on the same level, since the moments of R_h and R_h' about A and B are then equal to zero.

Equation (4) applies only to a load on the left segment, AC, of the arch. For a load on the right segment, a and b denoting, as before, the distances from the load to A and B respectively,

$$R_h=-R_h'=R_v\frac{l}{2h}=W\frac{b}{2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

For a series of vertical loads, the vertical components of the reactions may be found by summation. Thus

$$\Sigma R_v'=(Wa+W_1a_1+W_2a_2+\ .\ .\ .\ +W_na_n)\div l \quad . \quad . \quad (5)$$

$$\Sigma R_v\ =(Wb+W_1b_1+W_2b_2+\ .\ .\ .\ +W_nb_n)\div l \quad . \quad . \quad (6)$$

As a check, $\Sigma R_v+\Sigma R_v'=\Sigma W$. To facilitate the computation of the horizontal components, ΣR_h, of the reactions, it is desirable to apply Eqs. (5) and (6) separately to the loads on each segment of the arch. Then from Eqs. (4) and (4a)

$$\Sigma R_h=[\Sigma R_v'(\text{from loads on left segment}) \\ +\Sigma R_v(\text{from loads on right segment})]\frac{l}{2h}. \quad . \quad (7)$$

(*b*) *Single Horizontal Load. Graphic method.* In Fig. 11, the lines of action of R' and R are found as before. From D lay off $\overline{DE}=W$, and draw EF parallel to BD. Then $R=\overline{DF}$ and $R'=\overline{FE}$.

Algebraic method. From $\Sigma V=0$, $R_v=-R_v'$, and from $\Sigma M=0$, taking moments about A or B,

$$R_v=-R_v'=W\frac{c}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the known slope of R',

$$R_h' = R_v' \frac{l}{2h} = W \frac{c}{2h}, \qquad \text{(9)}$$

and from $\Sigma H = 0$,

$$R_h = W - R_h' = W\left(1 - \frac{c}{2h}\right) \qquad \text{(10)}$$

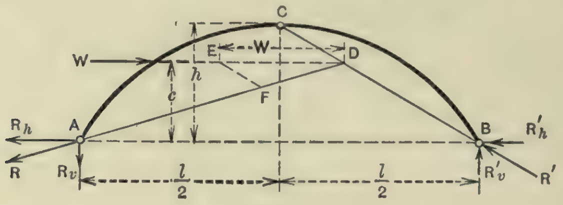


FIG. 11.

Note that R_h and R_h' both act in a direction opposite to W. If the direction of W is reversed, the directions of all the reactions will be reversed, but their magnitudes will be unaffected.

For a series of loads apply Eq. (8) to the individual loads, and combine in a manner similar to that shown in Eqs. (5) and (6). Then $\Sigma R_h' = (\Sigma R_v')\frac{l}{2h}$, and $\Sigma R_h = \Sigma W - \Sigma R_h'$.

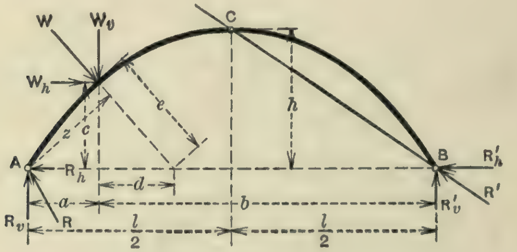


FIG. 12.

(c) *Single Inclined Load. Algebraically. First method.* Resolve the given load, W, Fig. 12, into its vertical and horizontal components, $W_v = Wc/e$ and $W_h = Wd/e$. Find the reactions from W_v and W_h separately, as above explained, and combine the two sets of values.

Second method. From $\Sigma M = 0$, taking moments about A,

$$R_v' = W \frac{z}{l} \qquad \text{(11)}$$

and from $\Sigma V=0$,

$$R_v = W_v - R_v' = W\left(\frac{c}{e} - \frac{z}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From the known slope of R',

$$R_h' = R_v'\frac{l}{2h} = W\frac{z}{2h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and from $\Sigma H=0$,

$$R_h = W_h - R_h' = W\left(\frac{d}{e} - \frac{z}{2h}\right) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The inclination of W and the height c being known, the values d, e and z may be either found by computation, or scaled from a large-scale drawing.

If the line of action of W intersects AB at a point between the supports, R_v acts upward. Again, if the intersection of R' and W occurs to the left of a vertical through A, or below AB, R_h acts towards the left, and R acts in the general direction shown in the figure.

Graphically. If W and R' are parallel, or nearly so, it is impracticable to determine the line of action of R by drawing a line from A to the intersection of W and R'. The reactions may then be found by means of an equilibrium polygon. This method will now be shown in its application to the most general conditions.

(*d*) *General Case. Graphic method.* Fig. 13 (*a*) represents the general case, in which (1) the supports are not on the same level, (2) the horizontal projections of the two segments are unequal, and (3) both sides are subjected to several loads of different inclinations.

Loads on One Segment. Let it be required to find the reactions for the loads *ab*, *bc* and *cd*, on the left segment of the arch. The line of action of the right reaction is then BC, and A is a known point on the other reaction. The reactions may therefore be found in precisely the same manner as in Fig. 6, Art. 24, the known line of action being now inclined instead of vertical.

In Fig. (*b*) draw the force polygon *abcda*. The desired reactions are components of the equilibrant *da*. Select a pole O_1 and draw the rays 1–4. In Fig. (*a*) construct the equilibrium polygon, beginning at A. Produce segment 4 to its intersection at D with BC, produced, and draw the closing line, 5, of the polygon. In Fig. (*b*) draw ray 5 parallel to this closing line to its intersection at m with dm, drawn parallel to BD. The required reactions, r_1 and r_1', are then respectively ma, acting from m to a, and dm acting from d to m.

The reactions, r_2 and r_2', from the loads on the right segment of the arch are found in precisely the same manner.

Loads on Both Segments. The reactions, R and R', for the loads on the entire arch are the resultants of the reactions for the loads on each segment considered separately. In Fig. (*b*) draw mO parallel to dn and nO parallel to dm. Then $\overline{Oa} = R$ and $\overline{gO} = R'$. These reactions are seen to be the components of ga, the equilibrant of all the loads, as they should be.

If the loads are all vertical, the trial poles, O_1 and O_2, may be conveniently chosen at a common point on, or in the vicinity of, a horizontal line bisecting the force polygon, which is then a vertical line.

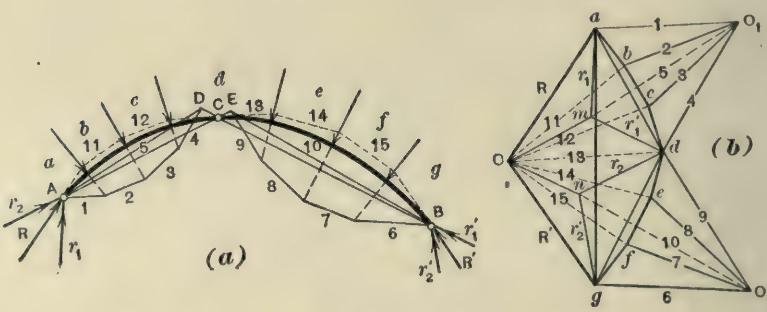


Fig. 13.

27. To Pass an Equilibrium Polygon Through Three Given Points. By a suitable location of the pole, an equilibrium polygon may be passed through any three (but not more than three) points, not lying in the same straight line. The construction in Fig. 13 furnishes a solution of this problem, A, B and C being the three given points and O being the required pole. This may be proved as follows:

First Proof. Since R and R' are the correct reactions of the arch, the resultants of the external forces to either side of C must be not only equal and opposite, but must pass through C.

In Fig. (*b*) draw the rays 11–15. In Fig. (*a*) produce R to its intersection with ab; then 11, drawn parallel to ray 11, is the position of their resultant, whose magnitude is given by ray 11. Continuing the construction of the equilibrium polygon, which is a resultant polygon for the forces acting on the arch, 12 is the position of the resultant of R, ab and bc; and 13, which must pass through C, is the position of the resultant of all external forces on either segment of the arch. The last segment, 15, of the equilibrium polygon must meet fg and R' at their point of intersection. The construction of the equilibrium polygon might have been begun at this point, or at C and continued in both directions.

Second Proof. In Fig. (*b*) the reactions r_1 and r_1' were found by assuming the pole at O_1, which led to the closing line AD in the equilibrium polygon, Fig. (*a*). If segment 4 of this polygon had passed through C, then AC would have been the closing line. The values of the reactions r_1 and r_1' will obviously be unaffected by the location of the pole. Therefore, since mO, Fig. (*b*), is drawn parallel to AC, a pole anywhere on this line will lead to the closing line AC, and thus cause the equilibrium polygon of the forces to the left of C to pass through C and A. Similarly, since nO is drawn parallel to BC, a pole anywhere on this line will cause the equilibrium polygon of the forces to the right of C to pass through C and B. Hence the pole O, which lies at the intersection of the lines mO and nO, will cause the equilibrium polygon of the entire force system to pass through A, B and C.

CHAPTER IV

SHEARS AND BENDING MOMENTS IN SIMPLE BEAMS

28. Definitions. A *beam* is a structure supported at one or more points, and designed to carry loads acting in directions perpendicular or oblique to its longitudinal axis. When the material is wood and the structure consists of a single piece it is usually called a 'joist' or a 'beam.' When the material is steel, the structure is usually called a 'beam,' if it consists of a single piece; a 'girder,' if it consists of several pieces riveted together into a solid whole; and a 'truss' if it consists of several pieces joined together by rivets or otherwise into an open framework designed, as a whole, to act as a beam. In practice the terms 'beam,' 'girder' and 'truss' are used loosely, and to a great extent, interchangeably. Thus in building construction a beam which serves wholly or partly to support other beams is called a girder, whether it consists of a single piece or of several pieces riveted together. On the other hand, in bridge construction riveted floor girders supporting other beams are usually called floor beams, while riveted trusses are frequently called riveted girders, or 'lattice' girders.

29. Conditions for Equilibrium. A *simple beam* is one supported at the ends in such a way as to render the reactions statically determinate. The external forces acting on the beam are the loads and the end reactions. Unless otherwise stated, the usual conditions are to be understood, namely, that the beam is horizontal, and that the external forces are vertical and coplanar. In order that the beam may be in equilibrium the external forces must satisfy the laws of equilibrium, i.e., in general, $\Sigma V=0$, $\Sigma H=0$ and $\Sigma M=0$, or for vertical forces only, $\Sigma V=0$ and $\Sigma M=0$.

30. Shear. The *shear* at any vertical section of a beam is the algebraic sum, i.e., the resultant, of all external forces to *either* side of the section. Upward forces are called positive and downward forces negative. Conventionally, the shear at a given section is said to be positive (+) or negative (−), according to whether the algebraic sum of the forces to the *left* of the section is positive (resultant upward), or negative (resultant downward). Thus in Fig. 1 (*a*) the shear, V_1, at section z_1, is positive when $R_1 > w_1 + w_2$ and negative when $R_1 < w_1 + w_2$.

Let W_1 and W_2 denote the sums of the loads to the left and right, respectively, of section z_1. Then $R_1 - W_1$ and $R_2 - W_2$, the algebraic sums of the forces to the left and right, respectively, of section z_1, must be equal in magnitude, but opposite in sign, since for equilibrium the algebraic sum of these sums must equal zero ($\Sigma V = 0$). In computing the shear at a section it is convenient, therefore, to find the algebraic sum of the forces on that side of the section on which they are fewer in number. If, however, the forces to the *right* are thus considered, the sign of the resultant must be *reversed* to conform with the conventional signs for shears.

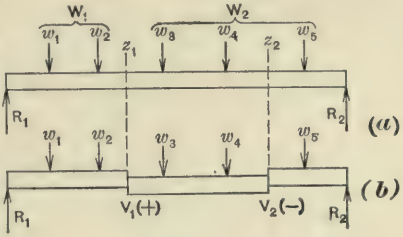


FIG. 1.

Since the resultants of the forces on opposite sides of any given section are equal in magnitude but opposite in direction, they tend to cause sliding at that section. Thus in Fig. (*a*), assume V_1 at section z_1 to be positive and V_2 at section z_2 to be negative. Then at section z_1, the parts to the left and right tend to slide up and down respectively, whereas at section z_2 the tendency is in the opposite direction.

31. Shear Diagram. The vertical shears throughout a beam may be represented by a diagram in which the ordinate at any section denotes to a convenient scale the shear at that section. Positive shears are indicated by ordinates above, and negative shears by ordinates below the horizontal datum line.

(*a*) *Concentrated Loads.* The shear diagram for a series of concentrated loads is bounded by horizontal lines with vertical offsets at the loads (Fig. 2), each offset representing to scale the magnitude of the corresponding load.

At load w_3 the shear passes through zero. To the left of that load the shear equals $R_1-(w_1+w_2)$, and to the right it equals $-(R_2-w_4)$. The *numeric* sum of these shears, $R_1+R_2-(w_1+w_2+w_4)$, equals w_3. It is incorrect to speak of the shear *at* a load, for immediately to the left of a load the shear has one value, and immediately to the right it has another value differing from the first by the magnitude of the load. Similarly, the ordinate R_1 in the shear diagram is, strictly speaking, not the shear *at* the left support, but *immediately to the right of* that support.

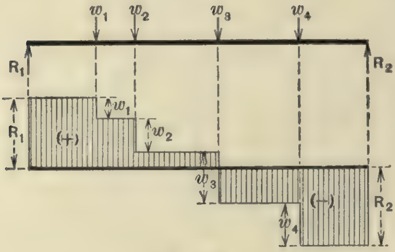


FIG. 2.

The (+) and (−) areas in the shear diagram must be equal, for considering the case of a single load W, Fig. 3, and taking moments about the point of application of the load ($\Sigma M=0$) $R_1a=R_2b$. Since the shear diagram for a series of concentrated loads may be regarded as a composite diagram of the shear diagrams for the individual loads the (+) and (−) areas must be equal in general.

(*b*) *Uniform Load.* The shear diagram for a uniform load covering the beam wholly or in part is bounded by inclined lines for the distances over which the load extends, as shown in Fig. 4. The slopes of these lines vary with the intensity of the load. For distances free from load, the boundary lines are horizontal, since the shear is constant for such distances (assuming the weight of the beam to be neglected as being relatively small, which is true in most cases).

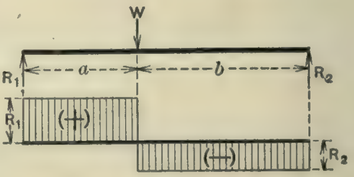


FIG. 3.

If a uniform load of intensity w covers the whole beam, as in Fig. (a), the shear at any section z, distant x from the left support, is

$$V=R-wx=w\left(\frac{l}{2}-x\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the equation of a straight line. For $x=0$, $V=\frac{1}{2}wl$; for $x=l$, $V=-\frac{1}{2}wl$; and for $x=\frac{1}{2}l$, $V=0$.

To construct a shear diagram, the reactions are first found, which may be done most conveniently by computation, and the shears at the extremities of the uniform loads are then derived. These values are laid off as ordinates, with due regard to sign, to any convenient scale (which bears, of course, no relation to the horizontal scale of distances), and their extremities are then connected by straight lines. For the usual case of downward loads, the inclined lines slope downward to the right, as in Fig. 4.

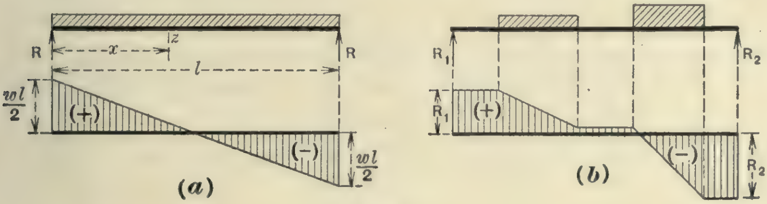


FIG. 4.

32. Bending Moment. The bending moment at any vertical section of a beam is the algebraic sum, i.e., the resultant, of the moments of all external forces to *either* side of the section about any point in that section. Clockwise moments are called positive and counter-clockwise moments negative. Conventionally, the bending moment at a given section is said to be positive (+) or negative (−), according to whether the algebraic sum of the moments of the forces to the *left* of the section is positive (resultant clockwise), or negative (resultant counter-clockwise). Thus a positive bending moment produces compression in the upper fibres of the beam and tension in the lower fibres, and, through the elastic deformation of the fibres, convexity downward, whereas a negative bending moment produces the opposite effects. For the usual case of loads acting downward a simple beam is convex downward for its entire length; hence the bending moment is positive at every section. This may also be shown mathematically as follows:

Referring to Fig. 5, let M denote the bending moment at any section. At any section to the left of W, the moment of R_1 is clockwise, hence M is positive. At any section to the right of W, the moment of R_2 is counter-clockwise, hence the resultant moment of the forces R and W, to the left of the section, must be equal in magnitude, but opposite in sign (clockwise) to the moment of R_2, since for equilibrium the algebraic sum of the moments to the right and left must equal zero ($\Sigma M=0$). Thus at any section to the right of W, M is also positive. Since for a single load the bending moment is positive throughout the beam, that is true also for any number of loads.

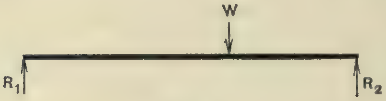


FIG. 5.

Since the resultant moments of the external forces to either side of a section are equal in magnitude but opposite in sign, it is convenient in computing the bending moment at a given section to find the algebraic sum of the moments on that side of the section on which the forces are fewer in number. If, however, the moments to the *right* are thus considered, the sign of the resultant moment must be *reversed* to conform with the conventional signs for bending moments.

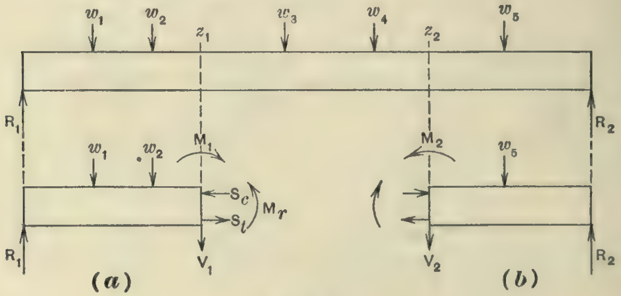


FIG. 6.

If a section is taken through a beam, as at z_1, Fig. 6, the internal forces—i.e., the stresses—at that section, Fig. (*a*), must hold in equilibrium the external forces to *either* side of that section (Art. 3). Let V_1 and V_2 denote the shears at section z_1 and z_2 respectively, and let it be assumed that V_1 is positive and V_2 negative. Since the part of the beam to the right of section z_1 tends to slide downward at that section, V_1 acts downward on the part of the beam to the left of the section shown in Fig. (*a*). Again, since the part of the beam to the left of section z_2

tends to slide downward at that section, V_2, Fig. (*b*), acts downward on that part of the beam to the right of section z_2.

Let S_c and S_t denote the resultants of the compressive and tensile fibre stresses at section z_1, then $(\Sigma H=0)$, $S_c=S_t$, and the moment M_r of this couple, called the *resisting moment*, must be equal to and opposite in direction to the positive bending moment, M_1, at the section. Since the bending moment at section z_2 is also positive, the moment M_2 of the external forces to the right of that section must be counter-clockwise, as shown in Fig. (*b*). The moment of the fibre stresses at that section is equal to and opposite in direction to M_2, the upper fibres being in compression and the lower in tension, as at section z_1.

Beams must be so designed that the internal stresses induced by the shears and bending moments will not exceed safe limits. The design of beams will be treated in Vol. III. This chapter will be devoted especially to the consideration of shears and bending moments in simple beams. Many of the principles involved find an immediate application, however, in the analysis of stresses in trusses, as will be seen later.

33. Relations between Shears and Bending Moments. Referring to Fig. 7, let W denote the resultant (in position as well as in magnitude) of all loads for the distance b from the left support, and w the intensity of the uniform load for the distance c. The bending moment at any section z along c is then

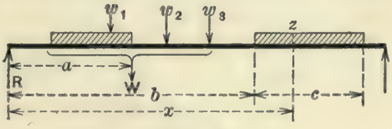


FIG. 7.

$$M=Rx-W(x-a)-w(x-b)\cdot\tfrac{1}{2}(x-b).$$

Differentiating

$$\frac{dM}{dx}=R-W-w(x-b)\ldots\ldots\ldots\quad(2)$$

To find the condition for maximum or minimum bending moment, let the first derivative dM/dx equal zero, whence

$$\frac{dM}{dx}=R-W-w(x-b)=0\quad\ldots\ldots\quad(3)$$

The second member of Eq. (2) is seen to represent the shear at the section in question, and dM/dx denotes the 'rate of change' of the bending moment. Hence the following important principles:

I. The rate of change of the bending moment at any section is equal to the shear at that section.

II. If the shear is zero at a section covered by a uniform load, the bending moment is maximum or minimum at that section.

III. If the shear changes from positive to negative—i.e., *passes through* zero—at a concentrated load, then (from I) the bending moment is maximum or minimum at the section under that load.

IV. If the shear is zero for any distance, the bending moment is constant for that distance.

In a simple beam the condition of zero shear is always indicative of *maximum* bending moment, as will be seen later.

34. Moment Diagram. The bending moments throughout a beam may be represented by a diagram, called briefly a 'moment diagram,' in which the ordinate at any section denotes to a convenient scale the bending moment at that section. Positive moments are indicated by ordinates above and negative moments by ordinates below the horizontal datum line.

(*a*) *Concentrated Loads.* If V denotes the shear at any section, then by Eq. (2), Art. 33,

$$\frac{dM}{dx} = V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is to say, the 'rate of change' of the bending moment, or the *slope* of the moment diagram at any section, is equal to the shear at that section. Since V is constant between consecutive loads the slope of the moment diagram must be constant for that distance. As V decreases, the slope decreases, and vice versa. When V is positive the slope is upward to the right, when V is negative it is downward to the right, and when V is zero, the slope is zero—i.e., the line is horizontal. Hence, the bending moment is maximum at the section where the shear is zero, or *passes through* zero.

The moment diagram, Fig. 8 (*a*), exhibits each of the above-mentioned characteristics in relation to the shear diagram, Fig. (*b*). Beginning at the left reaction, the slope of AB is upward to the right because the shear is positive. Since the shear is constant for each of the spaces a, a_1, a_2, etc., the slope is constant for these distances, and since the shear changes suddenly at each load, the slope undergoes a sharp change at B, C, D, etc. The positive shear decreases at w_1 and again at w_2, hence the slope becomes flatter to the right of B and C. At w_3 the shear passes through zero and becomes negative; therefore the slope to the right of D is downward and the bending moment is maximum at D. At w_4 the negative shear increases, and the slope becomes correspondingly steeper. If M_1 and M_2 denote the bending moments at w_1 and w_2, respectively, and if V_1 is the shear immediately to the right of w_1, then by Eq. (4), $V_1 = \tan \alpha$, and

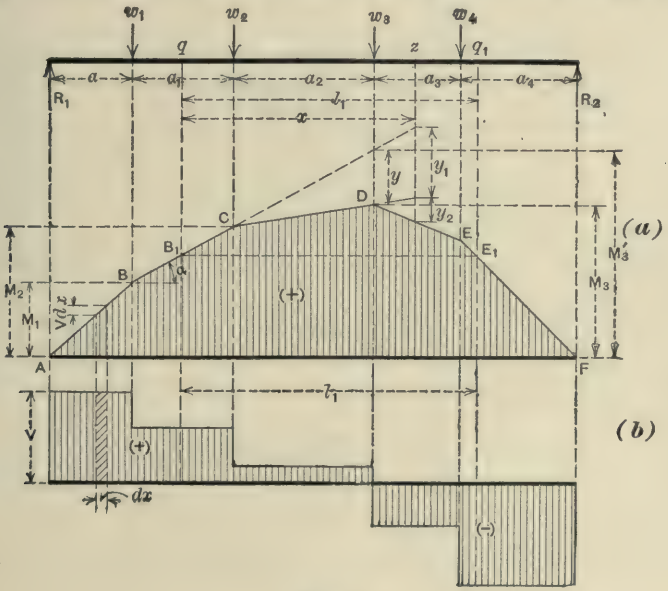


FIG. 8.

$$M_2 = M_1 + V_1 a_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence the principle:

v. The bending moment at any load is equal to the bending moment at the next load to the left, plus the product of the shear immediately to the right of the latter by the distance between the loads.

The moment of R_1 and w_1 about w_3 has the value

$$M_3' = M_1 + V_1(a_1 + a_2).$$

This moment is represented in Fig. (*a*) by the ordinate, at w_3, to BC produced. The moment of R_1, w_1 and w_2 about w_3—i.e., the bending moment at w_3—is

$$M_3 = M_3' - w_2 a_2.$$

Therefore the vertical intercept y, between BC produced and CD, equals the moment $w_2 a_2$, whence the principle:

> **VI.** The moment of any vertical force about any point is equal to the vertical intercept at that point between the lines in the moment diagram adjacent to the force.

Similarly the moments of w_2 and w_3 about any section z are equal to the intercepts y_1 and y_2 respectively. If M_q is the bending moment at the section q between the loads w_1 and w_2, the bending moment at any section z, between the loads w_3 and w_4, may therefore be expressed as follows:

$$M_z = M_q + V_1 x - (y_1 + y_2). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence the following general principle, which is an extension of Principle V:

> VII. The bending moment at any section is equal to the bending moment at any other section to the left, plus the product of the shear at the latter (or immediately to the right of the latter, if the section is taken *at* a load) by the distance between the sections, minus the sum of the moments of the loads between the sections about any point in the section at which the bending moment is sought.

The bending moment M_q at section q, Fig. 8, is equal to the moment of the resultants, $S_c = S_t$, of the fibre stresses, Fig. 9, and V_1 represents the shear at the section. The bending moment at z is therefore the algebraic sum of M_q and the moments of V_1, w_2 and w_3 about z, as expressed in Eq. (6).

Reverting to Fig. 8, if B_1E_1 is drawn horizontally, the bending moment at section q_1, vertically above E_1 and distant l_1 from section q, is also equal to M_q. If V_4 denotes the shear at section q_1, the forces at that section act as shown in Fig. 9. Since the end couples M_q are equal and opposite, the vertical forces alone are also in equilibrium. This can be true only if V_1 and V_4 are equal to the end reactions of a simple beam of length l_1 and loaded as shown. Thus in Fig. 8, the ordinates below B_1E_1 denote the bending moment M_q which is constant for that distance, and the ordinates above B_1E_1 represent the bending moments in an imaginary simple beam of length l_1 subjected to the same loading as exists for that distance on the actual beam. The portion of the shear diagram corresponding to the distance l_1 is obviously the shear diagram for the imaginary beam of that length.

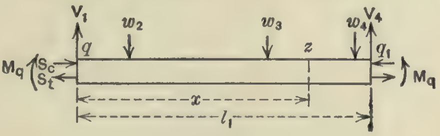


FIG. 9.

If M and V denote respectively the bending moment and shear at a given section, the bending moment at a second section, at a differential distance dx from the first, will be $M \pm Vdx$. But Vdx is the area of the shear diagram between ordinates at the extremities of dx, as shown in Fig. 8, and that will be true also if the shear diagram is bounded by an inclined line, as for uniform loading. Hence, in general,

> VIII. The bending moment at any section is equal to the area of the shear diagram to the left of the section, this area being regarded as the algebraic sum of plus and minus areas.

(*b*) *Uniform Load.* Referring to Fig. 10, the bending moment at any section z, within the distance covered by the uniform load of intensity w, has the value

$$M = R_1x - w(x-a)\cdot\tfrac{1}{2}(x-a),$$

or

$$M = R_1x - \tfrac{1}{2}w(x-a)^2. \qquad (7)$$

This equation is of the general form

$$y = \pm c_1 x^2 \pm c_2 x \pm c_3, \quad \text{. . . . . . .} \quad (8)$$

in which c_1, c_2 and c_3 are constants. From analytical geometry, Eq. (8) is the equation of a parabola; hence the bending moment diagram for distances covered by a uniform load is always bounded by a parabolic curve. Figs. (*a*) and (*b*) represent the shear and bending moment diagrams respectively. The vertex of the parabola lies at C, since the shear is zero at that section.

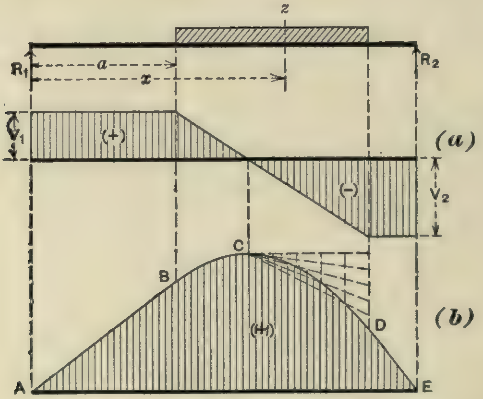


FIG. 10.

The slope of AB equals V_1 $(=R_1)$, and the slope of DE equals $V_2(=R_2)$. AB slopes upward to the right because V_1 is positive, DE downward to the right because V_2 is negative. The shears at sections at differential distances to the right and left of B differ by a differential. Hence AB is tangent to the parabola at B and similarly DE at D—that is to say, the slope of the moment diagram does not change suddenly at the extremities of a uniform load, as it does at concentrated loads.

To construct the bending moment diagram, locate the critical points B, C and D by laying off the computed bending moments as ordinates at corresponding sections. The longer arc, CD, of the parabola may then be constructed after locating points on the curve by the familiar method illustrated in the figure, and the arc BC may be drawn by symmetry.

For the usual condition of downward loads the general trend of the shear diagram between supports is downward to the right—that is to say, positive shears decrease and negative shears increase from left to right for the distances between supports. Hence

> IX. For downward loads the general curvature of the moment diagram is concave downward.

As has been seen, the bending moments for simple beams under downward loads are positive throughout, so that the moment diagrams for simple beams lie wholly above the horizontal line.

Note.—The nine principles enunciated in this and the preceding article are of fundamental importance in the analysis of shears and bending moments in beams and trusses. Their derivation and significance should be thoroughly mastered by the student. As a rule, the study of Mechanics of Materials precedes that of Framed Structures, so that the contents of this chapter up to this point are then to be regarded somewhat in the nature of a review for which illustrative numeric examples, which have been omitted, may be considered unnecessary.

The following practical problem involves the application in an interesting way of some of the principles in the relation of shears and bending moments. Its solution is left as an exercise to the student.

Problem. In building design a girder has frequently to be proportioned for a uniform load over its entire length, and a single concentrated load from a column or transverse beam. Prove that under these conditions:

1. The bending moment is maximum at the concentrated load or at a section between that load and the center of the girder.
2. That the distance from the center of the girder to the section at which the bending moment is maximum is the product of the concentrated load by its distance from the nearer support, divided by the total uniform load, provided the distance thus computed is less than that from the center of the girder to the concentrated load. If the computed distance is greater the bending moment is maximum at the concentrated load.

Note.—To solve this problem, draw typical shear diagrams for the uniform load and the concentrated load; then combine these diagrams and analyze algebraically.

35. Moment Diagram for Broken Uniform Loading. Fig. 11 represents a beam with broken or non-continuous uniform loading of different intensities. Let it be required to draw the moment diagram. The shear diagram is shown in Fig. (*a*). The shear is zero at *O*, and the bending moment is therefore maximum at that section. The vertex of the parabola *DEF*, Fig. (*b*), therefore lies at *E*. The points *B*, *C*, *D* and *E* may be determined by laying off the computed bending moments at the corresponding critical sections. The parabola *DEF* is constructed as in the preceding article. To draw the parabolic arc *BC*, the vertex, *G*, may be located as follows: The slope of this curve at any point is equal to the ordinate to *IK* at the corresponding section. Hence this line produced will determine the section *H* at which the slope of the parabola is zero, i.e., the vertex. The line *CD* is tangent to both parabolas. Hence the vertex *G* on the ordinate *GH* may be located from the property of the parabola that the vertex bisects the sub-tangent.

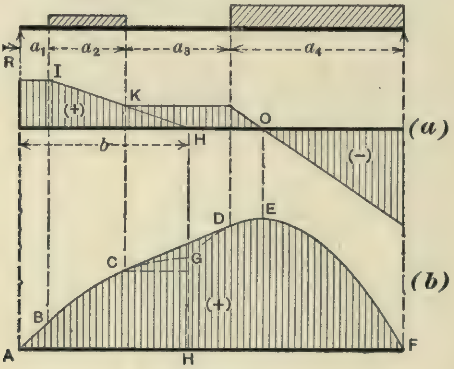


FIG. 11.

The distance b, from the left support to H, may be computed as being equal to $a_1 + R/w$, in which w is the intensity of the uniform load for the distance a_2. The ordinate GH equals $Rb - \frac{1}{2}w(b - a_1)^2$.

36. Beam Subjected to Inclined Forces. If one or more of the loads on a beam are inclined, the resultant of the horizontal components of the loads must be balanced by a horizontal reaction at one or both supports.

In Fig. 12 the right end is assumed to rest on rollers so that no horizontal reaction can be developed at that support. The left end is assumed to be fixed, but in such a way as to admit of angular movement in a vertical plane. The condition of free angular movement at the supports applies to simple beams in general, since otherwise the bending moments at the ends are not zero, as assumed, and the reactions are not statically determinate (Art. 23).

Let all the inclined forces, Fig. (*a*), be produced to their intersections, C, C_1, C_2, etc., with the longitudinal axis of the beam, and let the forces be resolved into vertical and horizontal components at these points of intersection, as shown in Fig. (*b*). The vertical reactions R_v and R_1 may then be found in the usual way from the vertical components, w_v', w_v'', etc., of

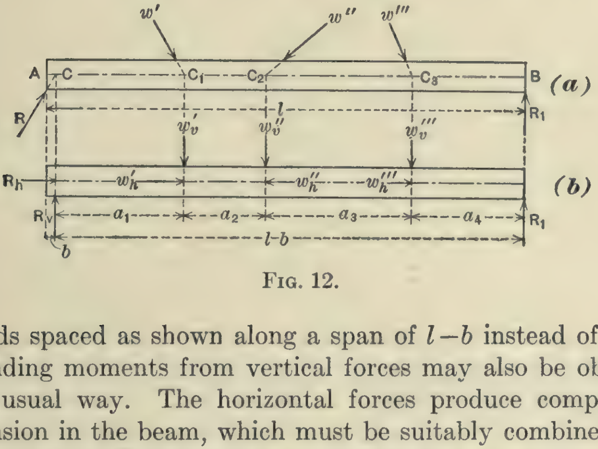


FIG. 12.

the loads spaced as shown along a span of $l-b$ instead of l, and the bending moments from vertical forces may also be obtained in the usual way. The horizontal forces produce compression and tension in the beam, which must be suitably combined with the fibre stresses from bending in proportioning the beam. Thus for a_1 the stress is $-R_h$; for a_2, $-(R_h+w_h')$; for a_3, $+w_h'''$; and for a_4, zero. In these expressions $R_h=w_h''-(w_h'+w_h''')$.

If the beam is fixed at both ends the horizontal reactions are statically indeterminate. The assumption is then usually made that the horizontal reactions at the two supports are equal.

37. Bending Moments by Graphic Methods. (*a*) *Inclined Forces.* In Fig. 13 (*a*) let AB represent the longitudinal axis of a beam. Assume the left support fixed, and the right support on rollers. The reactions R and R_1 from the loads *ab*, *bc*, *cd*, etc., may be found graphically as shown in Fig. (*b*), in the manner explained in connection with Fig. 6, Art. 24. Let it be required to find the bending moment at any point, as, for example, q.

First method. The bending moment at q is equal to the resultant moment of R and load ab about q—i.e., it is equal to the moment of their resultant, hb, Fig. (b), acting from h to b, about q. The position of this resultant, r, in Fig. (a) is found by drawing a line parallel to hb through C, the intersection of segments 2 and 8, to either side of the forces R and ab (Art. 19). Through q draw DE parallel to r, and from C drop the perpendicular z. Then the desired bending moment is $M_q = rz$. In Fig. (b) drop the perpendicular H_1 on bh produced; H_1 is the pole distance of the resultant hb ($=r$). From the similar triangles CEF and Obh, $\overline{hb} \times z = H_1 \times \overline{EF}$. Letting i_1 denote the intercept EF between segments 2 and 8 adjacent to the forces whose moment about q is required,

$$M_q = H_1 i_1. \qquad \qquad (9)$$

Similarly the bending moment at u from the six forces R, ab, bc, cd, de, and ef, to the left, is $M_u = H_2 i_2$.

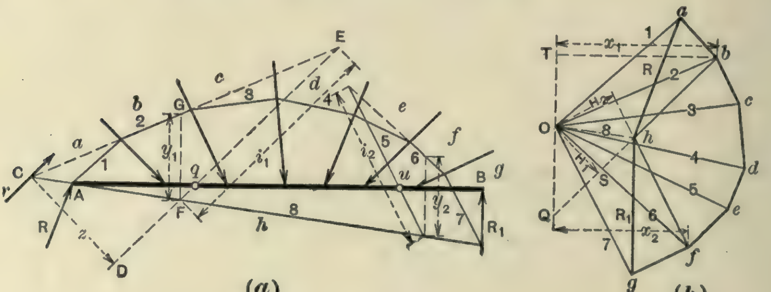


FIG. 13.

In finding the bending moment at any point the moment of one of the reactions about that point is always involved; therefore the closing line, 8, of the equilibrium polygon is always one of the segments between which the intercept is measured. Thus, in general, to find the bending moment at any point, proceed as follows:

1. Draw a line through the given point, parallel to the

resultant (in the force polygon) of the forces to the left of the point.

2. Measure the intercept on this line between the closing line of the equilibrium polygon, and the other segment adjacent to the forces to the left.
3. The required bending moment is the product of this intercept by the pole distance of the resultant.

If the resultant (in the force polygon) of the forces to the left of the given point should be coincident with ray 8 produced, the line in the equilibrium polygon parallel to this resultant will be parallel to the closing line and to the other segment adjacent to the forces. The length of the intercept is then infinity, and the pole distance of the resultant is zero, so that this method becomes inapplicable. Again, if the resultant is nearly parallel with the ray 8, as, for example, the resultant *hd*, of *R*, *ab*, *bc* and *cd*, the angles in the equilibrium polygon, between the line parallel to the resultant and the two segments between which the intercept is measured, are so acute that the method becomes impracticable. In such cases the following method may be used:

Second method. Let it be required to find, for example, the bending moment $M_q = H_1 i_1$ by another method.

Through F, Fig. 13 (*a*), draw the vertical FG; through O, Fig. (*b*), draw the vertical QT, meeting the horizontal from b at T and the line bh produced at Q.

From similar triangles EFG and bQO,

$$\frac{i_1}{y_1} = \frac{Qb}{QO}.$$

From similar triangles QTb and QOS,

$$\frac{x_1}{H_1} = \frac{Qb}{QO}.$$

Equating the first members of these equations,

$$H_1 i_1 (= M_q) = x_1 y_1. \qquad \ldots \ldots \quad (10)$$

It may be shown similarly that $M_u = x_2 y_2$. Thus, in general, to find the bending moment at any point, proceed as follows:

1. Draw a line from the given point, parallel to the resultant (in the force polygon) of the forces to the left of

the point, to its intersection with the closing line of the equilibrium polygon.

2. From this point of intersection draw a vertical line to its intersection with the other segment (produced, if necessary) of the equilibrium polygon adjacent to the given forces.
3. The product of this vertical by the horizontal projection of the ray to the force polygon, parallel to the segment in the equilibrium polygon to which the vertical was drawn, is then the desired bending moment.

If the resultant of the forces should be coincident with ray 8 produced, the required vertical in the equilibrium polygon is then the vertical distance between the closing line and the other segment adjacent to the forces, which is in that case parallel to the closing line.

Third method. If the intersections of the given forces are sufficiently favorable, the simplest method is to determine the resultant of the forces whose moment about a given point is desired and to find then the moment of this resultant about the given point.

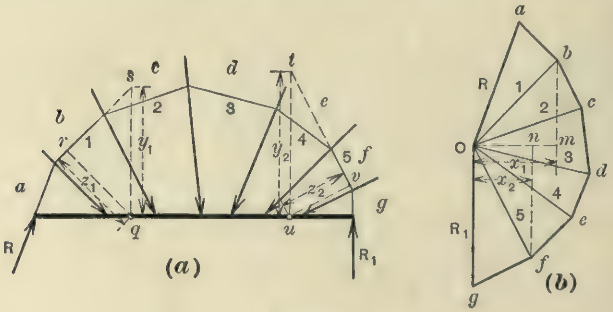


FIG. 14.

Thus in Fig. 14, after the reactions R and R_1 have been found graphically, or by computation, choose the pole O, Fig. (*b*), at the intersection of the reactions. Then each segment in the equilibrium polygon, Fig. (*a*), which in this case is a resultant polygon, indicates the position of the resultant of all external forces to either side of that segment. The magnitude of each resultant is given by the corresponding ray in Fig. (*b*).

From q and u drop the perpendiculars qr and uv to segments 1 and 5, respectively; and in Fig. (*b*) drop the perpendiculars *bm* and *fn* to a horizontal through *O*. Then if M_q and M_u denote, as before, the bending moments at *q* and *u*, $M_q = \overline{Ob} \times z_1$ and $M_u = \overline{Of} \times z_2$. From the similar triangles *qrs* and *Obm*, $z_1/y_1 = x_1/Ob$. Therefore

$$M_q = x_1 y_1. \qquad \text{(11)}$$

It may be shown similarly that $M_u = x_2 y_2$. Thus, in general, *the bending moment at any point is the product of the vertical distance from that point to the resultant of the external forces to either side of that point, by the horizontal projection of the ray in the force polygon parallel to that resultant.*

(*b*) *Vertical Forces.* For the usual case of vertical forces the procedure is greatly simplified. Referring to Fig. 15, the reactions *R* and R_1 may be most readily found by computation. Their values are laid off in the force polygon, Fig. (*b*), as *fg* and *ga* respectively. By selecting the pole, *O*, at some point on a horizontal through *g*, the closing line, 7, of the equilibrium polygon, Fig. (*a*), will be horizontal. The equilibrium polygon is drawn in the usual manner, beginning on a vertical through either support. Following now the three concluding directions for the general cases by either the first or second method, it is seen that if M_q and M_u denote the bending moments at *q* and *u*, respectively, $M_q = Hy_1$ and $M_u = Hy_2$. Thus, in general, *for vertical forces the bending moment at any point is the product of the intercept in the equilibrium polygon, on a vertical through that point, by the pole distance.* By laying off the pole distance as some even unit value, e.g., 100, 1,000, 10,000, etc., to the scale used in the force polygon, the bending moments may be scaled

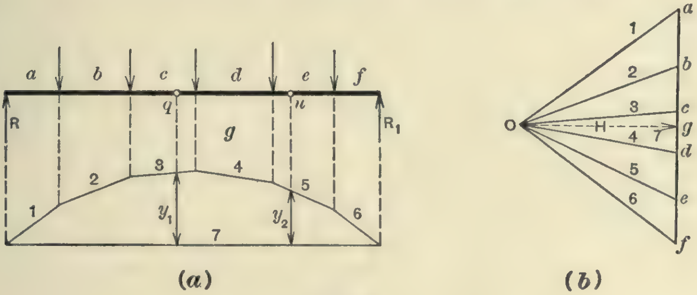


FIG. 15.

off directly, as y_1, y_2, etc., in terms of the unit value chosen for H. It is thus seen that a bending moment diagram (Art. 34), is in fact an equilibrium polygon based on a pole distance of unity.

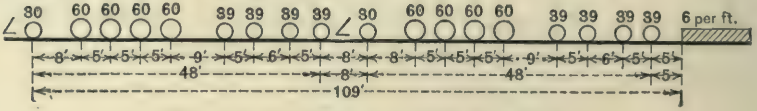
38. Live-Load Systems. The loads to which a beam is subjected consist of (*a*) its own weight and other permanent weight, known as the *dead* load, and (*b*) extraneous loads, known as *live* loads. The dead load is usually a uniformly distributed one. The live loads for which beams are commonly designed may be classified as follows:

(*a*) A series of concentrated loads of fixed magnitude and spacing.
(*b*) A uniform load.
(*c*) A series of concentrated loads followed by a uniform load.
(*d*) A uniform load with superadded concentrated loads.

Occasionally, though rarely, the live load is a continuous load of varying intensity.

A uniform load may be regarded as a special form of concentrated-load system. Thus if its intensity is w, it is equivalent to a series of equal concentrated loads, $w \cdot dx$, separated by differential distances dx.

In building design the position of the live load is usually definitely fixed, whereas in bridge design the live load is assumed to occupy that position in which its effect on the particular function under consideration—usually stress or deflection—is maximum.



Two 213-ton Engines + 6000 lbs. per ft.

FIG. 16.—COOPER'S E-60 LOADING.

In American practice railroad bridges are commonly designed for load systems consisting of a series of axle loads representing two locomotives coupled, followed by a train load which is assumed to be uniformly distributed. The load systems prescribed in Cooper's specifications for railroad bridges are used more extensively than any others. Cooper's E–60 Loading is represented in Fig. 16. The figures above the wheels are the axle loads in thousands of pounds. The designation " E–60 " indicates that the driver-axle concentrations, which are the heaviest, weigh 60,000 lbs., and that the train load weighs 6000 lbs. per foot. The length of each engine (with tender) is seen to be 48 ft., and the distance from the pilot wheel of the first engine to the head of the train load, 109 ft.

In all of Cooper's load systems, ranging usually from E–30 to E–60, the wheel spacing is the same, and all loads in any one system bear a fixed ratio to the corresponding loads in every other system, that ratio being expressed by the ratio of the figures in the designation of the systems. Thus the loads in Cooper's E–40 Loading may be found by multiplying those in Fig. 16 by 40/60.

39. Moment Table for Cooper's E–60 Loading. The computation of reactions, shears, bending moments and moments in general, is greatly facilitated by moment tables from which the moment of any series of consecutive loads about the first load to the right of the series, as well as the sums of loads and distances, may be read at a glance.

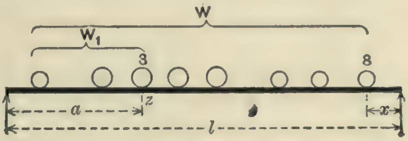


FIG. 17.

In Fig. 17 let it be required to find the bending moment at section z under wheel 3. Let M denote the moment of loads W about wheel 8, and m the moment of loads W_1 about wheel 3. The bending moment at z is then

$$M_z = (M + Wx)\frac{a}{l} - m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which the values M, m and W may be read from the moment table.

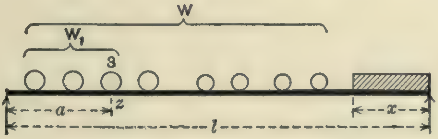


FIG. 18.

If the load is partly uniform, as in Fig. 18, let M denote the moment of loads W about the head of the uniform load, and w

the intensity of the latter. Then with the notation otherwise as before,

$$M_z = (M + Wx + \tfrac{1}{2}wx^2)\frac{a}{l} - m,$$

or

$$M_z = [M + (W + \tfrac{1}{2}wx)x]\frac{a}{l} - m, \quad . \quad . \quad . \quad . \quad (13)$$

in which the values of M, m and W may be read from the moment table.

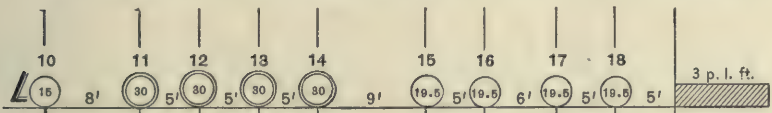
A moment table for Cooper's E–60 Loading, in separate form, accompanies this book. Its construction will be explained by reference to Fig. 19, which shows only a single engine—i.e., wheels 10 to 18, inclusive. The figures within the circles denote the wheel loads—not axle loads—in thousands of pounds. They are therefore the loads borne by one of a pair of beams, girders, or trusses in a single-track bridge. The figures between the loads indicate the successive intervals in feet, which are laid off to a convenient scale.

The principal figures in the table denote moments in thousands of foot-pounds. Thus the values in line (*a*) are the moments of any series of consecutive loads from wheel 18 to the left, about the head of the uniform load. For example, the moment about that point of loads 12 to 18, inclusive, is 4160. Similarly the values in line (*b*) are the moments of any series of consecutive loads from wheel 17 to the left about wheel 18. Thus the moment of loads 14 to 17, inclusive, about wheel 18 is 1375. In like manner, to find the moment of loads 11 to 14, inclusive, about wheel 15 follow the vertical line through wheel 15 downward to the heavy stepped line, then continue toward the left to the vertical line through wheel 11, and read the moment, 1980, to the right of the latter. The moments about the two middle drivers 12 and 13 are carried out in both directions from the stepped line. Thus the moment of loads 14 to 17, inclusive, about wheel 13 is 1280. The moments to the right of the stepped line are convenient in finding the maximum loads on floor beams, as will be seen later (Art. 45).

In these tabulated moments, only as many significant figures are retained as are needed to keep the error within a limit of 0.1 per cent., and the last of these significant figures is made 5 whenever this does not increase the error beyond the limit stated.

The figures in the upper left-hand corners denote the sums of the various series of successive loads. Thus to find the sum of loads 11 to 16, inclusive, follow the vertical through wheel 16 to the stepped line, continue toward the left to the vertical line through wheel 11 and read the sum, 159.0, in the space to the right of the latter. Similarly the figures in the upper right-hand corners denote the sums of distances. Thus the distance between wheels 11 and 16 is 29 ft., which is found in precisely the same manner as just explained for the summation of loads. Again, the distance from wheel 12 to wheel 18 is 35 ft.

Referring now to the separate moment table for two engines,



	10	11	12	13	14	15	16	17	18
(*a*)	53 6310	45 5510	40 4160	35 2965	30 1915	21 1015	16 605	10 292.5	5 97.5
(*b*)	213.0 48 5240	198.0 40 4520	168.0 35 3325	138.0 30 2275	108.0 25 1375	78.0 16 624	58.5 11 312	39.0 5 97.5	
(*c*)	193.5 43 4280	178.5 35 3630	148.5 30 2580	118.5 25 1680	88.5 20 932	58.5 11 331.5	39.0 6 117		
(*d*)	174.0 37 3230	159.0 29 2680	129.0 24 1808	99.0 19 1088	69.0 14 518	39.0 5 97.5			
(*e*)	154.5 32 2460	139.5 24 1980	109.5 19 1260	79.5 14 690	49.5 9 270				
(*f*)	135.0 23 1245	120.0 15 900	90.0 10 450	60.0 5 150					
(*g*)	105.0 18 720	90.0 10 450	60.0 5 150	60.0 5 150	79.5 14 423	99.0 19 794	118.0 25 1280	138.0 30 1865	
(*h*)	75.0 13 345	60.0 5 150	60.0 5 150	90.0 10 450	109.5 19 820	129.0 24 1289	148.5 30 1875	168.0 35 2555	
(*i*)	45.0 8 120								

FIG. 19.

it is to be noted that when only a single engine is wholly or partly on the bridge, the desired values are to be read from the right half of the table. Thus the moment of loads 2 to 7, inclusive, about wheel 8 is the same as the moment of loads 11 to 16, inclusive, about wheel 17, i.e., 3630. The division between the first and second engine is distinguished by a heavy vertical line.

The values in the upper horizontal line are the moments of any series of consecutive loads from wheel 2 to the right, about wheel 1. Thus the moment of loads 2 to 14, inclusive, about wheel 1 is 14,400. These moments are occasionally serviceable in computations.

The vertical lines through the loads are continued to the upper edge of the paper. By laying off a given span and its panel divisions on a horizontal line to the same scale as that used for the moment table (1 in. =15 ft.), and applying the upper edge of the latter to such a line, the wheels which are on the bridge when any given wheel is placed at any desired point may be seen at a glance.

The application of the moment table to simple operations of frequent occurrence will be illustrated by the following examples:

Example 1. Locate the center of gravity of wheel-loads 8 to 17, inclusive.

Solution. The moment of loads 8 to 17, inclusive, about wheel 17 is 6360. To find this value, follow the vertical line through wheel 17 to the stepped line, and continue toward the left to the vertical line through wheel 8. In the same subdivision of the table the sum of loads 8 to 17, inclusive, is found to be 232.5. The distance from the center of gravity of the loads in question to wheel 17 is then 6360/232.5= 27.36 ft. The distance from wheel 17 to wheel 13 is 25 ft. Hence the center of gravity lies 2.36 ft. to the left of wheel 13.

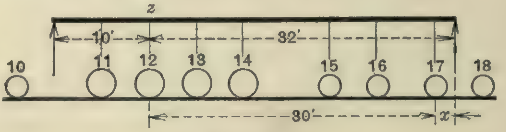


FIG. 20.

Example 2. Find the live-load reaction at the left support of a girder whose span is 42 ft. between centers of supports, when wheel 12 is 10 ft. from the left support.

Solution. Lay off the span, 42 ft., on a straight line to a scale of 1 in. = 15 ft., mark the section z, 10 ft. from the left support, and apply the upper edge of the moment table to this line with wheel 12 at z, as shown in Fig. 20. Wheels 11 to 17, inclusive, are seen to be on the girder. From the moment table the distance from wheel 12 to wheel 17 is found to be 30 ft. Therefore $x = 32 - 30 = 2$ ft. With the same notation as in Eq. (12), $R = (M + Wx)/l$, in which the values M and W may be read from the moment table as explained. Thus the left reaction, in thousands of pounds, is,

$$R = (3630 + 178.5 \times 2) \div 42 = 94.9.$$

Note.—After a little practice in the use of the moment table it is not necessary to lay off the span to scale. For the simple conditions involved in this example the above expression for R may easily be written directly from inspection of the moment table.

Example 3. For the conditions stated in Example 2, compute the live-load bending moment at wheel 14.

Solution. Wheel 14 lies 10 ft. to the right of wheel 12 and therefore 20 ft. from the left support. The moment, 900, of loads 11 to 13 about wheel 14 is read from the moment table. From the preceding example, $R = 94.9$. The desired bending moment in thousands of foot-pounds is therefore

$$M_{14} = 94.9 \times 20 - 900 = 998.$$

Example 4. Required the live-load bending moment in a girder of 50 ft. span between centers of supports at a section 16 ft. from the left support when wheel 14 is at that section.

Solution. The section is 34 ft. from the right support. From the moment table, wheel 14 is 30 ft. from the head of the uniform load; therefore the distance x from the right end covered by the uniform load is 4 ft. The distance from wheel 14 to 11 is 15 ft.; hence wheel 11 is the last wheel on the span toward the left. The moment of loads 11 to 18, inclusive, about the head of the uniform load is 5510 and the sum of those loads is 198. The moment of loads 11 to 13, inclusive, about wheel 14 is 900. Hence the desired bending moment, in thousands of foot-pounds, is by Eq. (13):

$$M_{14} = [5510 + (198 + \tfrac{1}{2} \times 3 \times 4)4]\frac{16}{50} - 900 = 1124.$$

40. Maximum Shears from Live Loads. The usual problem is to compute the maximum shear at one or more sections in a beam, from a given load system. In a beam the *character* of the shear—positive or negative—is seldom or never of importance.

(*a*) *Uniform Load.* Referring to Fig. 21, a load to the right of any section z produces a positive shear at that section equal to the left reaction from that load. A load to the left of z produces a negative shear at that section, since the load is necessarily greater than the left reaction from that load. Hence, for maximum positive shear at any section the load must extend from the right support to that section. If w denotes the intensity of the uniform load, the maximum positive shear at z is

$$V = \frac{w}{2l}x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is the equation of a parabola, whose vertex is at the right support as shown in Fig. (*a*). For $x = l$, $V = \frac{1}{2}wl$, and for $x = \frac{1}{2}l$, $V = \frac{1}{8}wl$.

For the load advancing from the left, the negative shear at any section becomes maximum when the head of the load reaches that section. The shear, *irrespective of sign*, becomes maximum at any section when the load extends from that section to the *farther support.* The maximum shears therefore vary from $\frac{1}{8}wl$ at the center of the beam to $\frac{1}{2}wl$ at the ends.

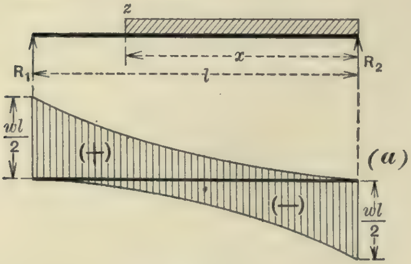


FIG. 21.

(*b*) *Single Concentrated Load.* If a load W is at any section z of a beam, Fig. 22, the shear $V_l(=R)$ immediately to the left of the section is Wx/l, and the shear $V_r(=R-W)$, immediately to the right of the section is $-W(l-x)/l$. Thus it is seen that as the load advances from a position immediately to the right of a section to one immediately to the left, the shear at that section changes suddenly from $+Wx/l$ to $-W(l-x)/l$. The magnitude of the change is therefore the algebraic difference between these quantities—i.e., W, as shown in the figure.

The shaded part of Fig. 22 is the shear diagram for W in the fixed position shown. The equation $V_l=Wx/l$ is that of a straight line. For $x=0$, $V_l=0$ and for $x=l$, $V_l=W$. As W advances over the span from right to left, the point E in the shear diagrams for successive positions will therefore describe the straight line BC whose ordinate AC at the left support equals W.

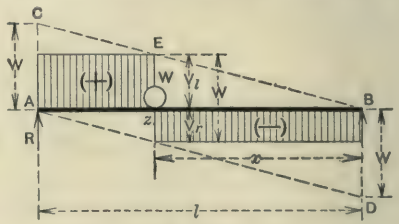


FIG. 22.

(*c*) *Series of Loads. End shear.* Referring to Fig. 23, let W denote the sum of all loads on the beam, and x the distance of their center of gravity from the right support. The shear at a section immediately to the right of the left support is then

$$V=R=\frac{Wx}{l}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

It is seen, therefore, that the end shear increases both with W and x, and that it is greatest when their product is maximum. The position which satisfies this requirement must be found by trial. It can usually be determined from a mere inspection of the load system. The left end shear from engine loading is always maximum when one of the end drivers is at a section immediately to the right of the left support, and the right end shear is maximum when one of these drivers is at a section immediately to the left of the right support. If the center of gravity of the drivers is at the center of the group, as is usually the case, or if it is between the center and the first driver, the end shear is usually maximum when the loading occupies the position shown in Fig. 23, with the first driver immediately to the right of the left support. If, on the other hand, the center of gravity, g, of the drivers is back of the center, i.e., when $a<b$, Fig. 24, the end shear is usually maximum when the loading is situated as in Fig. 24, with the last driver immediately to the left of the right support. In that case the value Wx in Fig. 24 is usually greater than its value in Fig. 23, although that depends to some extent also on the relative magnitude and spacing of the other loads. Thus for Cooper's E–60 Loading, in which the center of gravity of the drivers is at the center of the group, the position in Fig. 24 governs for spans from 23 to 27.3 ft., and that in Fig. 23 for all other spans from 10 to 100 ft. It is true in general, however, that for the engines headed toward the left, the end shear is maximum either at the left end for the load placed as in Fig.

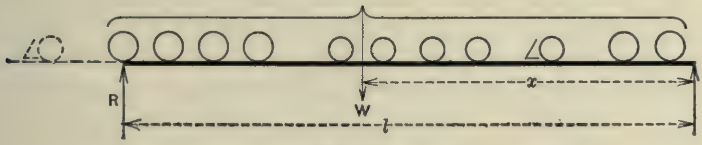


FIG. 23.

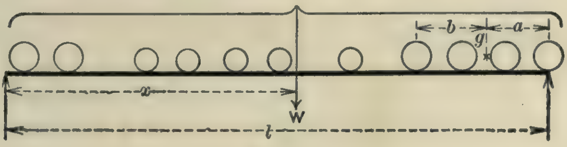


FIG. 24.

23 or at the right end for the load placed as in Fig. 24; and vice versa for the engines headed toward the right.

Shear at Any Section. Let the condition for maximum positive shear at a section z, Fig. 25, be considered. When the loading is in the position shown, with w_1 immediately to the right of z, the shear at that section is

$$V_1 = R = \frac{Wx}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If the loading is advanced a differential distance to the left and w_1 is thus brought immediately to the left of z, the shear at that section is suddenly decreased by w_1—that is to say, the shear at z is now $Wx/l - w_1$. If the loading is advanced further toward the left the lever arm x increases progressively, and likewise R and the shear at z. When the loading has been advanced a distance a_1, and w_2 is now immediately to the right of z the shear at that section is

$$V_2 = R - w_1 = \frac{W(x+a_1)}{l} - w_1. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

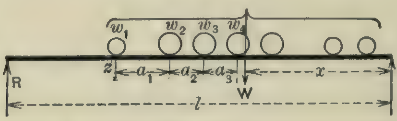


FIG. 25.

Comparing this equation with Eq. (16) it is seen that $V_2 > V_1$ when $W > \frac{l}{a_1} w_1$. Similarly, if V_3 is the shear at z when w_3 is immediately to the right of that section, $V_3 > V_2$ when $W > \frac{l}{a_2} w_2$.

Thus the following criteria for maximum positive shear at z are established:

w_1 governs when $W >$ zero, and $< \frac{l}{a_1} w_1$,

w_2 governs when $W > \frac{l}{a_1} w_1$, and $< \frac{l}{a_2} w_2$,

w_3 governs when $W > \frac{l}{a_2} w_2$, and $< \frac{l}{a_3} w_3$, etc.

Since a load to the left of z produces negative shear at that section, it is obvious that for maximum positive shear at z, the loading to the left must be relatively small or zero. Thus if the loads are all equal and equally spaced, w_1 will always govern, since W is then less than lw_1/a_1 for every section except the one immediately adjacent to the end support, even though as many loads as possible are assumed to be brought upon the span as the load advances toward the left. For the section adjacent to the end support, $W = lw_1/a_1$, and, since $w_1 = w_2 = w_3$, etc., and $a_1 = a_2 = a_3$, etc., the criterion for each load is satisfied at that section, as it should be under the conditions stated, which approximate that of a uniform load.

In general, to ascertain whether the criterion for any particular load w_1, w_2, w_3, etc., is satisfied, that load must be placed at (immediately to the right of) the section. The criteria are correct, in a comparative way, only if W, the total loading on the beam, is constant. In shifting the loading to bring successive wheels to the section in question certain loads may pass off the span, and other loads may come on. In either case the value of W will change unless it should so happen that when a load passes off at one end of the span an equal load comes on at the other end. Under changing values of W, the criteria for two successive wheels may both be fulfilled. In that case the shear must be computed for each position to determine which is greater.

For a given loading, W, the probability that w_2 will govern rather than w_1 is increased as w_1 becomes relatively small and a_1 relatively large. These conditions both apply to the pilot wheel of an engine, and therefore for engine loading, w_2—i.e., the first driver, almost invariably governs. Thus for Cooper's E–60 Loading, $w_1 = 15$, $w_2 = 30$, $a_1 = 8$ and $a_2 = 5$. Hence w_2 governs when $W > 1.87\,l$ and $< 6\,l$. If for the loading advancing from the right, that criterion is fulfilled when w_2 is placed at any section from the center of the beam to the left end, w_2 will govern similarly for the shears throughout the right half of the beam for the loading reversed and advancing from the left end.

Thus it is seen that w_2 will govern when the *average load*, W/l, on the beam, in thousands of pounds per foot, is greater than 1.87 and less than 6. When w_2 is placed at the center of the span, the average load per foot is always greater than 1.87. When w_2 is placed immediately to the right of the left support, the average load can be greater than 6 only in case the span is so short that only the drivers come upon the beam. The load system is then reduced to a system of four equal loads equally spaced, in which case, as has been pointed out, the first load of the series, i.e., w_2, will always govern. Thus it is seen that *the shear at any section in a beam from Cooper's Loading is maximum when the first driver is placed at that section with the following loads extending from that section to the farthest support.*

The only exception to this rule is to be found in the circumstance already pointed out that for spans from 23 to 27.3 ft. the negative shears in the vicinity of the right end of the beam are slightly greater when the *fourth* driver is placed immediately to the left of the section (as in Fig. 24 for the end section), than the positive shears at corresponding sections in the left half of the beam when the *first* driver is placed immediately to the right of the section (as in Fig. 23 for the end section). This will be further considered in the example at the end of this article.

For positive shears in the right half of the beam w_1 governs when $W < 1.87l$. These shears are obviously smaller than the maximum negative shears at the same sections when the loading is reversed and extends over the greater segment of the span to the left, with w_2 immediately to the left of the section. Since the design of beams is influenced only by the magnitude of the shears, irrespective of their character, an analysis to determine the exact distance for a given span for which each wheel governs is of no practical value. An analysis of this kind applicable to trusses will be given later (Art. 116).

Example. A bridge consists of a pair of girders, 25 ft. long, between centers of end supports, bearing cross-ties which are assumed to form a continuous floor. Find the maximum live-load shears at the ends, the quarter points and the center of each girder from Cooper's E–60 Loading.

Solution. As stated in this article, for spans from 23 to 27.3 ft., with Cooper's Loading headed toward the left, the maximum negative shears in the vicinity of the right end of the girder are slightly greater than the maximum positive shears at corresponding sections in the left half of the girder.

At Ends. Placing the first driver, wheel 2, immediately to the right of the left support, the shear in the left end section, as found with the aid of the moment table, is

$$V_l = (1980 + 139.5 \times 1) \div 25 = 84.8.$$

Placing the fourth driver, wheel 5, immediately to the left of the right support, the shear in the right end section may be found by subtracting the left reaction from the total load on the beam. Thus

$$V_r = 135 - (1245 \div 25) = 85.2.$$

This value is seen to be slightly greater than that previously found.

At Quarter Points. Placing wheel 2 at a section 6.25 ft. from the left end,

$$V_l = (900 + 120 \times 3.75) \div 25 = 54.0.$$

If wheel 5 is placed 6.25 ft. from the right end, the loading will be equal in magnitude and symmetrical as to its distribution, in comparison with the conditions above assumed. Therefore $V_r = V_l$.

At Center. Placing wheel 2 at the center of the girder,

$$V_l = (720 + 105 \times 2.5) \div 25 - 15 = 24.3.$$

Placing wheel 5 at the center, the shear at a section immediately to the right of the center may be found by subtracting the left reaction from the sum of the loads to the left of the section. Thus

$$V_r = 90 - (1260 + 109.5 \times 3.5) \div 25 = 24.3.$$

This value is seen to be equal to that previously found.

41. Diagram of Maximum Shears from Cooper's E–60 Loading. The values of the maximum shears at the ends, quarter-points and center of a girder (two girders assumed) from Cooper's E–60 Loading, are shown as ordinates to the heavy lines in Fig. 26, for spans from 10 to 100 ft. If the same end shear, V_e, for any given span, l, is produced by a uniform load of intensity w, $V_e = \frac{1}{2}wl$. The maximum shear at the quarter point from that load would be $(\frac{3}{4}wl \times \frac{3}{8}l)/l = \frac{9}{32}wl = \frac{9}{16}V_e$, and the maximum shear at the center would have the value $(\frac{1}{2}wl \times \frac{1}{4}l)/l = \frac{1}{8}wl = \frac{1}{4}V_e$. The ordinates representing these shears would thus be in the ratio $1 : \frac{9}{16} : \frac{1}{4} :: 16 : 9 : 4$. For any given span the ordinates to the light lines, for quarter point and center shears, are laid off in the above ratio to the ordinate which represents the maximum end shear for that span. The greater shears at the quarter points and center from Cooper's Loading as compared with the shears from uniform loads which produce the same end shears as Cooper's Loading may thus be judged by inspection.

For spans ranging from 25 to 100 ft. the curves represented by the heavy lines are seen to be nearly coincident with the straight lines shown dotted. The equations of these straight lines, from which the maximum shears may be computed with slight errors, or which may be used to check values obtained by more accurate means, are as follows:

For maximum shear at ends,

$$V = 1.9\,l + 35. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

For maximum shear at quarter points,

$$V = l + 30 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

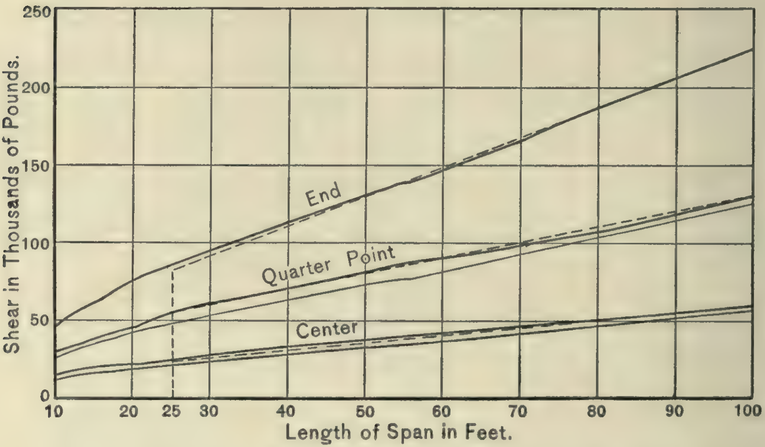


FIG. 26.—Maximum Shear in One Girder (Two Girders Assumed) from Cooper's E–60 Loading.

For maximum shear at center,

$$V = 0.5\,l + 10 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

A table of maximum end shears from Cooper's E–40 Loading for spans ranging from 10 to 125 ft., is appended to Cooper's "General Specifications for Steel Railroad Bridges."

The maximum end shear as expressed in Eq. (18) is seen to be that produced by a uniform load of intensity 3.8 covering the entire beam, and a concentrated load of 35 acting at the end of the beam.

The equivalent uniform load per track which produces the same maximum end shear as Cooper's E–60 Loading ranges from 6800 lbs. per foot for a 25-ft. span, to 4500 lbs. per foot for a 100-ft. span.

42. Maximum Bending Moment at a Given Section. (*a*) *Uniform Load.* Since a load at any point of a simple beam produces positive bending moments at every section, the bending moments throughout the beam will be maximum when the uniform load extends over the entire beam. If w denotes the intensity of the uniform load, the maximum bending moment at any section z, Fig. 27, is

$$M = \tfrac{1}{2}wlx - wx \cdot \tfrac{1}{2}x$$

or

$$M = \tfrac{1}{2}wx(l-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, *the maximum bending moment from a uniform load at any section of a simple beam is equal to the intensity of the load multiplied by one-half the product of the two segments of the span.*

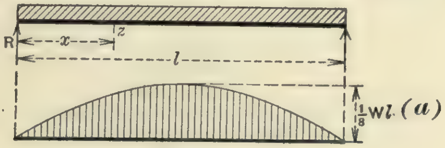


FIG. 27.

Eq. (21) is that of a parabola, Fig. (a), whose vertex lies above the center of the beam. If W denotes the total load on the beam, then for $x = \tfrac{1}{2}l$,

$$M_{\max} = \tfrac{1}{8}wl^2 = \tfrac{1}{8}Wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the design of beams it is frequently convenient to express the maximum bending moment in terms of inch-pounds rather than foot-pounds. In that case the value l in the expression $\tfrac{1}{8}Wl$ should be reduced to inches. Beginners frequently make the error of entering the expression $\tfrac{1}{8}wl^2$ with w in pounds per foot, and l in inches. The result is then 12 times too great, since w should then be expressed in pounds per inch.

It is sometimes convenient to express the maximum bending moment at any section of a beam loaded uniformly, in terms of the distance, a, of the section from the center of the beam. Letting $x = \tfrac{1}{2}l - a$ and reducing, Eq. (21) becomes

$$M = \tfrac{1}{8}w(l^2 - 4a^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

(b) *Single Concentrated Load.* The moment diagram for a single load at any section z, is shown in Fig. 28 (a). The bending moment is maximum at z, where it has the value

$$M = \frac{Wx}{l}(l-x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If the same load were uniformly distributed over the entire beam its intensity would be W/l, whence from Eq. (21),

$$M = \frac{Wx}{2l}(l - x).$$

Comparing this equation with Eq. (24), it is seen that *the bending moment at any section is twice as great from a concentrated load at that section, as from the same load uniformly distributed over the whole beam.*

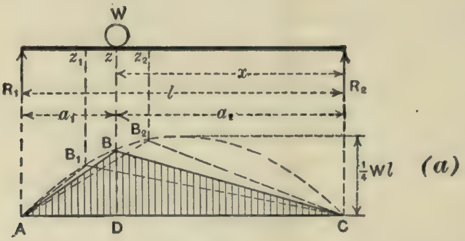


FIG. 28.

Eq. (24) is that of a parabola, Fig. (a), whose vertex lies above the center of the beam, where $x = \frac{1}{2}l$ and

$$M_{\max} = \tfrac{1}{4}Wl. \qquad (25)$$

If the load W is placed at a section z_1 to the left of z, or at z_2 to the right, the corresponding moment diagrams are AB_1C and AB_2C, respectively. The ordinate at z in either of these triangles is shorter than BD. Hence *the bending moment from a single concentrated load at any section of a simple beam is maximum when the load is placed at that section.*

This may also be shown as follows: When W is at z, $M_z = R_1a_1 = R_2a_2$. If W is moved to the left of z, $M_z = R_2a_2$, but R_2 is now smaller than before; similarly, if W is moved to the right of z, $M_z = R_1a_1$, but R_1 is now less than at first.

(c) *Series of Loads.* Referring to Fig. 29, the bending moment at any section z, between w_3 and w_4 for the loading in the position shown, has the value

$$M = Ra - W_1x_1.$$

If the loading is moved a distance b_1 to the right, so that w_3 is at z, the bending moment at that section is

$$M_1 = \left(R - \frac{Wb_1}{l}\right)a - W_1(x_1 - b_1).$$

It is seen that $M_1 > M$ when $Wa/l < W_1$. If this condition is not fulfilled it shows that the bending moment at z will be increased if the loading is moved a distance b_2 to the left, so that w_4 will be at z. Thus it is seen that *the bending moment at any section from a series of concentrated loads can become maximum only when one of the loads is placed at that section.*

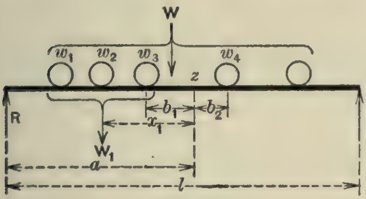


FIG. 29.

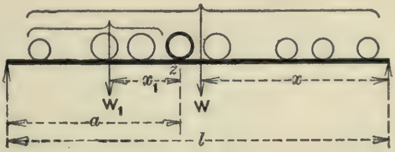


FIG. 30.

The exact position for maximum bending moment at any section z, Fig. 30, may now be found as follows: Placing one of the loads at z the bending moment at that section is

$$M = \frac{Wx}{l}a - W_1x_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

To find the condition for maximum M, let the first derivative dM/dx equal zero, noting that $dx = dx_1$, since any differential change in x from a differential movement of the loading will be accompanied by an equal differential change in x_1.
Hence

$$\frac{dM}{dx} = \frac{Wa}{l} - W_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

or

$$W = \frac{l}{a}W_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

This equation expresses the ‘ criterion ’ for maximum bending moment at any section distant a from the left support. The criterion is satisfied if, when the load at z is assumed to act a differential distance, first to the right, and then to the left of z, the second member of Eq. (28) is first less, and then greater, than the first member; for mathematically this implies a transition through the condition of equality.* This is equivalent to assuming the load at z to be divided into two parts, the one applied immediately to the left, the other immediately to the right of that section, in such proportions as to satisfy the criterion. The load which when placed at z satisfies the criterion is called the 'critical load.'

Eq. (28) may be written $W/l = W_1/a$, which signifies that the bending moment at z is maximum when the loading is so distributed that the average load over the distance a equals the average load over the entire span. For a uniform load, that condition is met for every section when the load extends over the whole beam, which is in that case obviously the condition for maximum bending moment at every section.

The critical wheel for any given section must be found by trial. This can be readily done with the aid of the moment table by laying off the span length on a straight line to the same scale as the wheel spacing in the moment table, and applying the upper edge of the table to such a line with some trial load at the section in question. The number of loads on the whole span, and over the distance a, may be seen by inspection and the sums, W and W_1, of such loads may be read from the table. For engine loadings, the bending moment at any section is usually maximum when one of the heavy drivers is placed at that section. If the criterion is satisfied by more than one position of the load, the bending moment should be computed for each position to determine its maximum value.

After the position which satisfies the criterion has thus been found, the corresponding bending moment may be obtained by Eq. (26), in which the moments Wx and W_1x_1 may be readily derived with the aid of the moment table, as explained in Art. 39.

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* If in Eq. (26) W_1 as well as W were a constant the equation would be that of a *single* straight line, and M would increase or decrease indefinitely (for equal increments in x and x_1) according to whether Wa/l is greater or less than W_1. In that case the condition for maximum M cannot be found by letting $dM/dx = 0$. W_1 is, however, not a constant, but it is the sum of all the loads to the left of z. Eq. (26) is therefore the equation of a *series* of straight lines whose slope, dM/dx, changes as W_1 changes, assuming W to remain constant. If in Eq. (27), $Wa/l > W_1$, dM/dx is positive, and if $Wa/l < W_1$, dM/dx is negative. Hence M is maximum at a given section when a differential movement of the load at the section, alternately to the right and left of that section, causes dM/dx to change from a positive to a negative value.

For a section at the center of the beam, a in Eq. (28) equals $\frac{1}{2}l$, whence $W = 2W_1$. After the position which satisfies this criterion has been found the corresponding bending moment may be conveniently computed (if a moment table is not available) as the sum of the products obtained by multiplying each load by one-half its distance from the *nearer* support. Thus, referring to Fig. 31, the bending moment at z from w is $wa/l \times \frac{1}{2}l = w \times \frac{1}{2}a$. Similarly the bending moment at z from w_1 equals $w_1 \times \frac{1}{2}a_1$, provided $a_1 < \frac{1}{2}l$.

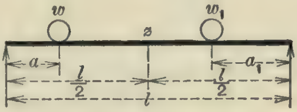


FIG. 31.

Example 1. Required the maximum bending moment from Cooper's E-60 Loading in a girder 55 ft. long between centers of supports, at a section 22 ft. from the end.

Solution. Lay off 55 ft. on a straight line to the same scale as that of the moment table. Apply the latter with wheel 12 at the section 22 ft. from the left end. Note that wheels 9 to 17, inclusive, whose sum is 213, are on the span. In the criterion, Eq. (28), W_1 equals 64.5 or 94.5 according to whether wheel 12 is assumed to act immediately to the right or left of the section. Since

$$213 > \frac{55}{22} \times 64.5 \quad \text{and} \quad < \frac{55}{22} \times 94.5$$

the criterion is satisfied.

The distance between wheels 12 and 17 (read from moment table) is 30 ft., so that wheel 17 is 3 ft. to the left of the right support. From the moment table, the moment of loads 9 to 16, inclusive, about 17 is 5270, and that of loads 9 to 11 inclusive about 12 is 755. Therefore the desired bending moment, in thousands of foot-pounds, is, by Eq. (26),

$$M = (5270 + 213 \times 3)\frac{22}{55} - 755 = 1609.$$

Note.—After a little practice in the use of the moment table it will not be found necessary to lay off the span to scale, since the conditions can be readily analyzed by a mere inspection of the moment table.

Example 2. Required the maximum bending moment, from Cooper's E-60 Loading, at the center of a girder having a span of 37 ft. between centers of supports.

Solution. First Method: With moment table. Place wheel 12 at center. Wheels 10 to 14 inclusive, are on the span. $W = 135$ and $W_1 = 45$ or 75.

$$135 > 45 \times 2 \quad \text{and} \quad < 75 \times 2.$$

The criterion is therefore satisfied. Wheel 14 is 8.5 ft. from the right support. The desired bending moment in thousands of foot-pounds is then

$$M = (1245 + 135 \times 8.5)\frac{1}{2} - 345 = 851.$$

Try wheel 13 at center. Wheels 10 to 15, inclusive, are on the span. $W = 154.5$, and $W_1 = 75$ or 105,

$$154.5 > 75 \times 2 \quad \text{and} \quad < 105 \times 2.$$

The criterion is therefore satisfied. Wheel 15 is 4.5 feet from the right support. The bending moment is

$$M = (2460 + 154.5 \times 4.5)\frac{1}{2} - 720 = 858.$$

This is greater than the value previously found, and it is the maximum bending moment at the central section.

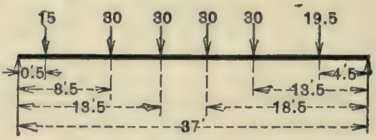


FIG. 32.

Second Method: Without moment table. After finding that wheel 13 satisfies the criterion for maximum bending moment at the central section, indicate on a sketch as in Fig. 32 (not drawn to scale) the distance of each load from the nearer support.

The bending moment at the center may be found as follows by the summation of products:

$$M = 15 \times \frac{0.5}{2} + 30 \times \frac{8.5 + 13.5 + 18.5 + 13.5}{2} + 19.5 \times \frac{4.5}{2} = 858.$$

43. The Greatest Possible Bending Moment in a Beam from a Concentrated Load System. The greatest possible live-load bending moment in a beam occurs in general at a section in the vicinity of the center of the span, but not at the center. For engine loading the greatest bending moment takes place at some section near the center when one of the drivers of the first or second engine, to either side of the center of gravity of the total loading on the beam, is placed at that section.

Referring to Fig. 33, the bending moment from the loading shown attains its greatest possible value when wheel w_1 or w_2, to either side of the center of gravity of the loading, is placed at a section whose location remains to be found. Let M_1 denote the bending moment at w_1, and m_1 the moment of the loads to the left of w_1 about that wheel. Then

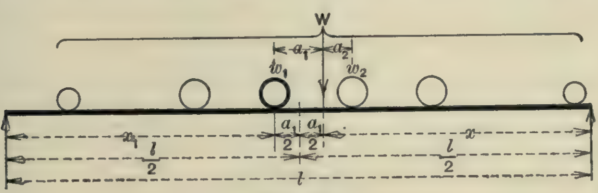


FIG. 33.

$$M_1 = \frac{Wx}{l}x_1 - m_1.$$

In this equation x and x_1 are the only variables, and since their sum, $x+x_1(=l-a_1)$, is constant, their product, and therefore M_1, is maximum when $x=x_1=\frac{1}{2}(l-a_1)$, in which case

$$M_1 = \frac{W(l-a_1)^2}{4l} - m_1. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

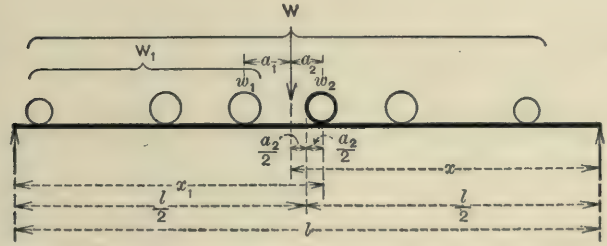


FIG. 34.

Similarly in Fig. 34, if M_2 denotes the bending moment at w_2, and m_2 the moment of the loads to the left of w_2 about that wheel:

$$M_2 = \frac{Wx}{l}x_1 - m_2,$$

in which M_2 becomes maximum when $x = x_1 = \frac{1}{2}(l + a_2)$, in which case

$$M_2 = \frac{W(l+a_2)^2}{4l} - m_2. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Thus it may be shown in general that *the bending moment at any given wheel becomes maximum when the distance from that wheel to one support equals the distance from the center of gravity of the loading to the other support*, or, otherwise stated, *when the center of the span bisects the distance from the center of gravity of the loading to the wheel in question.*

The bending moments M_1 and M_2 may be computed by Eqs. (29) and (30) respectively, without reference to the corresponding values of x, provided that when the loading is placed in the correct position on the span (laid off to scale on a straight line) the loading W on the span is found to agree with that for which the center of gravity was originally computed.

It is frequently assumed that M_1 or M_2 is greater according to whether w_1 or w_2 lies nearer the center of gravity of the loading. This is, however, not true in general. A criterion by which it may be determined in advance whether M_1 or M_2 is greater will now be derived.

From Eqs. (29) and (30), it is seen that $M_1 > M_2$, i.e., w_1 governs, when

$$\frac{W(l-a_1)^2}{4l} - m_1 > \frac{W(l+a_2)^2}{4l} - m_2.$$

Referring to Fig. 34, let W_1 equal the sum of the loads to the left of w_2. Then $m_2 = m_1 + W_1(a_1 + a_2)$. Substituting this value for m_2 and reducing (observing that a general change of signs reverses the sign of inequality), it is found that wheel w_1 governs when

$$W_1 > \left(\frac{1}{2} - \frac{a_1 - a_2}{4l}\right)W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

If this criterion is not satisfied, wheel w_2 governs.

For Cooper's loading the distance between successive drivers is 5 ft. Hence, for this loading, $(a_1 - a_2) < 5$ and the fraction $(a_1 - a_2)/4l$ is therefore relatively small compared with $\frac{1}{2}$. Unless W_1 is very nearly equal to $\frac{1}{2}W$, criterion (31) need not be solved accurately, but it may be seen by inspection whether it is or is not satisfied.

Example. Required the greatest bending moment in a girder whose span is 37 ft. between centers of supports, from Cooper's E–60 Loading.

Solution. Lay off 37 ft. on a straight line to the same scale as the moment table, and apply the upper edge of the latter with the drivers of the second engine near the center of the span, and in such a position as to bring as many loads as possible on the beam. Let it be judged that wheels 10 to 15, inclusive, will be on the span when the loading is placed in the correct position for maximum bending moment. The distance from the center of gravity of the loading to wheel 15 is $2460 \div 154.5 = 15.92$ ft. Hence the center of gravity lies between wheels 12 and 13, and at a distance $15.92 - 14 = 1.92$ ft. from wheel 13 and 3.08 ft. from wheel 12. Apply the load to the span so that the center of the latter lies first approximately $\frac{1}{2}$ of $3.08 = 1.54$ ft. to the right of wheel 12 and then $\frac{1}{2}$ of $1.92 = 0.96$ ft. to the left of wheel 13. Note that in either position the assumed loading is on the beam. If for the loading in one position one of the wheels had passed off the span or another wheel had come on, or both, the bending moment corresponding to the other position should be computed, and the process should then be repeated for the new loading to ascertain whether that would produce a greater bending moment.

Substituting in the criterion (31) it is seen that the wheel to the left of the center of gravity, in this case wheel 12, governs when

$$75 > \left(\frac{1}{2} - \frac{3.08 - 1.92}{4 \times 37}\right) 154.5.$$

It is seen by inspection that this criterion is not satisfied; hence wheel 13 to the right governs, and Eq. (30) is applicable. The maximum bending moment in thousands of foot-pounds is therefore

$$M_2 = \frac{154.5\,(37 + 1.92)^2}{4 \times 37} - 720 = 861.$$

This value is seen to be somewhat greater than the maximum bending moment (858) at the center of the beam, as found in Example 2, Art. 42.

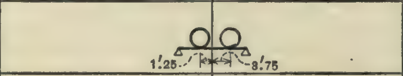
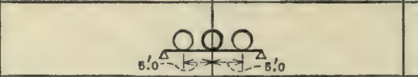
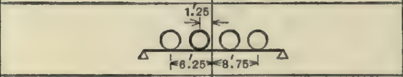
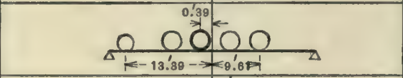
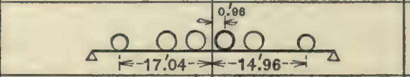
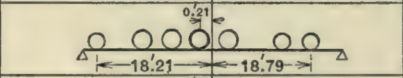
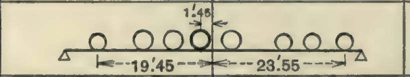
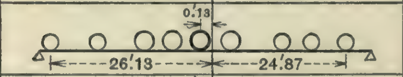
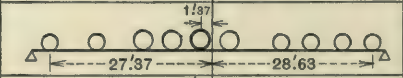
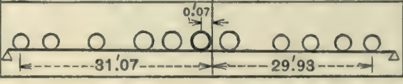
44. Special Formulas for the Greatest Possible Bending Moment in a Beam from a Given Concentrated Load System. By assigning suitable values to the constants, c_1, c_2 and c_3, Eqs. (29) and (30) may be expressed as follows:

$$M = c_1 l + \frac{c_2}{l} - c_3. \qquad \ldots \ldots \qquad (32)$$

The sign of c_3 is always negative; for, from Eq. (29), $c_3 = -\frac{1}{2}Wa_1 - m_1$, and from Eq. (30), $c_3 = \frac{1}{2}Wa_2 - m_2$, in which $\frac{1}{2}Wa_2$ is always less than m_2.

Convenient formulas for maximum bending moments may be derived for any given load system by finding the values of the constants in Eq. (32) for various groups of consecutive loads, and by determining the span limits within which each formula governs. Such data for Cooper's E–60 Loading are given in the following table, which is modeled, with the addition of some minor features, after tables originated by P. L. Szlapka.

SPECIAL FORMULAS FOR MAXIMUM BENDING MOMENTS FROM COOPER'S E–60 LOADING.

Case	Ctr. line of span	Span Limits		Max. Bending Mom. per Girder (2 Girders Assumed)
		Extreme Range	For Max. Bdg. Mom.	
(a)		7′.5 to 17′.5	8′.5 to 11′.1	$15l+\frac{94}{l}-75$
(b)		10′.0 to 20′.0	11′.1 to 18′.7	$22.5l-150$
(c)		17′.5 to 30′.5	18′.7 to 27′.7	$30l+\frac{188}{l}-300$
(d)		26′.8 to 37′.2	27′.7 to 35′.0	$33.75l-397$
(e)		34′.1 to 39′.9	35′.0 to 38′.6	$38.63l-568$
(f)		37′.6 to 49′.6	38′.6 to 48′.6	$43.5l-756$
(g)		47′.1 to 54′.9	48′.6 to 53′.8	$48.38l-993$
(h)		52′.3 to 59′.7	53′.8 to 58′.4	$53.25l-1255$
(i)		57′.3 to 64′.7	58′.4 to 63′.2	$58.13l-1540$
(k)		62′.1 to 70′.0	63′.2 to 70′.0	$63l-1848$

The vertical line which extends through all the diagrams indicates the position of the center of the span in each case relative to the critical load, distinguished by a heavy circle. The critical load is determined in each case by applying criterion (31), Art. 43. The given distances from the critical loads to the centers of the spans represent one-half the distances from the critical loads to the centers of gravity of the loadings. In two cases, (*g*) and (*i*), the critical load is seen to be the one farther removed from the center of gravity.

In the 'extreme range' column are given the limits of span within which each group of loads can be placed upon a beam in the correct position relative to its center without having one of the end wheels pass off, or having an additional wheel come on the span. The lower limit is twice the distance from the center of the beam to the *farthest* wheel *on* the span. Thus in case (*f*), $37.6 = 2 \times 18.79$. The upper limit is twice the distance from the center of the beam to the *nearest* wheel *off* the span. Thus in case (*f*), $49.6 = 2\ (18.79 + 6)$, which is less than the distance $2\ (18.21 + 8)$ to the first wheel off the span at the other end.

The next column contains the span limits within which each of the formulas for maximum bending moments, which appear in the last column, governs. These limits are found by equating the successive formulas and solving for l. Thus equating the formulas for cases (*e*) and (*f*):

$$38.63\,l - 568 = 43.5\,l - 756,$$

whence,
$$l = 38.6.$$

This value of l is therefore the upper limit of span within which the formula for case (*e*) gives a greater bending moment than that for case (*f*), and it is also the lower limit for case (*f*). It sometimes happens that the limiting values thus found lie outside of the practicable limits given in the preceding column. In that case the latter must, of course, control.

The formulas in the last column are derived from Eq. (29) or Eq. (30), according to whether the critical load lies to the left or right of the center line. Thus in case (*f*) Eq. (29) governs, i.e.,

$$M_1 = \frac{W(l - a_1)^2}{4l} - m_1,$$

in which $a_1 = 0.21 \times 2 = 0.42$, and from the moment table, $W = 174$ and $m_1 = 720$. Substituting these values in the above equation and reducing,

$$M_1 = 43.5\,l + \frac{7.7}{l} - 756.5.$$

The numerator of the second term, i.e., the constant c_2 in Eq. (32), equals $Wa_1^2/4 = 7.7$. In the table, this term is neglected whenever that can be done with an inaccuracy of less than 0.1 per cent.* Thus, finally, $M_1 = 43.5\,l - 756$.

For spans exceeding the highest value, 70.0, for case (*k*), the uniform train load will enter upon the span, so that the last equation is then no longer applicable. For spans exceeding that limit the greatest bending moment may be found with slight error by assuming it equal to the maximum bending moment at the center of the beam. The critical load is then determined by the criterion, $W = 2W_1$, derived from Eq. (28), Art. 42, and the bending moment may be found with the aid of the moment table, as in Example 2 (First Method) in that article.

Example. Required the maximum bending moment, from Cooper's E–60 Loading, in a girder having a span of 37 ft. between centers of support.

Solution. The span, 37 ft. falls between the limits 35.0 to 38.6 ft. for case (*e*). Hence by the equation for that case, the maximum bending moment in thousands of foot-pounds is

$$M_{\max} = 38.63\,l - 568 = 861.$$

This value is seen to agree with that found in the example of the preceding article, as it should.

45. Maximum Floor-Beam Loads. In highway bridges the live load is usually communicated to the transverse floor beams through longitudinal joists. The spacing of these joists depends on the character of the flooring and the magnitude of the live loading. If the joists are 2 to 3 ft. apart it is sufficiently accurate to regard the load on the floor beam as uniformly distributed. If the bridge is to carry electric railway traffic the tracks are usually carried by longitudinal steel girders, called *stringers*. Such stringers are commonly designed for a concentrated load system which represents the wheel spacing of an electric car. This loading is then also used to determine the maximum loads brought upon the floor beams by the stringers.

In railroad bridges the live loads are imparted to the floor beams through the track stringers, the load on each track being usually carried by a pair of stringers. The stringers and floor beams in a railroad bridge are designed for a concentrated load system.

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* The middle term is also dropped whenever the maximum inaccuracy can be kept within the limit of 0.1 per cent by modifying the last term. In every case, the error is reduced to a *minimum average* for the range of spans to which each formula applies.

The ends of the stringers are usually riveted to the floor beams. In computing the reactions of the stringers it is assumed, however, that they act like simple beams supported at the ends without constraint, although, strictly speaking, the actual conditions in the case stated are somewhat analogous to those for a beam continuous over a series of supports. The spaces between successive floor beams are called *panels.*

When the live load is uniform the floor-beam loads are of course maximum when the two panels adjacent to the floor beam are wholly covered by the live load.

The floor-beam loads from a concentrated load system become maximum when the loading occupies a certain position now to be determined.

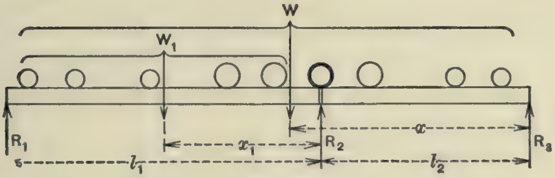

FIG. 35.

(*a*) *Criterion for Maximum. Unequal panels.* In Fig. 35 the panel lengths, l_1 and l_2, are unequal. It is required to find the position of the load system for which the load on the intermediate floor beam, which load is equal in magnitude and opposite in direction to the reaction R_2, becomes maximum.

Since the resultant load W and the three reactions from that load constitute a system of forces in equilibrium, their moment about any point must equal zero. Taking moments about the right support,

$$R_1(l_1 + l_2) + R_2\,l_2 - Wx = 0.$$

It is also seen that the first member of this equation denotes, by definition, the bending moment at R_3, whose value must be zero on the assumption premised, that the stringers act like simple beams. Substituting for R_1 its value W_1x_1/l_1,

$$R_2 = \frac{W}{l_2}x - \frac{W_1(l_1+l_2)}{l_1l_2}x_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

To find the condition for maximum R_2 let the first derivative equal zero,* whence

$$W = \frac{l_1 + l_2}{l_1} W_1, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

which is the criterion for the maximum floor-beam load. In seeking the position which will satisfy this criterion, the critical load should be placed at R_2, since a differential movement of the load, alternately to the left and right of that point, will then affect W_1 while W remains constant. The criterion is fulfilled when the conditions explained in connection with Eq. (28), Art. 42, are met.

The criterion for maximum floor-beam load R_2, Eq. (34), is seen to be the same as the criterion for maximum bending moment at R_2 in a simple beam of span $l_1 + l_2$, Eq. (28), Art. 42.

Equal panels. For the usual case of equal panels, $l_1 = l_2 = l$, and Eq. (34) becomes

$$W = 2W_1. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

This criterion is always fulfilled with one, and usually with both of the middle drivers as the critical load.

(*b*) *Maximum Load. First method.* In Eq. (33) let $W_1x_1 = M_1$ and $Wx = M_2$. Then

$$R_2 = \frac{M_2}{l_2} - \frac{M_1(l_1 + l_2)}{l_1 l_2},$$

or

$$R_2 = \frac{M_2 - M_1}{l_2} - \frac{M_1}{l_1}, \quad . \; . \; . \; . \; . \; . \; . \quad (36)$$

in which the values of M_1 and M_2 are readily found with the aid of a moment table.

For equal panels, $l_1 = l_2 = l$, and Eq. (36) becomes

$$R_2 = \frac{M_2 - 2M_1}{l}. \quad . \; . \; . \; . \; . \; . \; . \quad (37)$$

Second method. Referring to Fig. 35,

$$R_2 = W - (R_1 + R_3).$$

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* See footnote, Art. 42, page 74.

Let M_1 denote as before the moment of the loads in panel l_1 about R_2, and let M' denote the moment of the loads in panel l_2 about the same point. Then

$$R_2 = W - \left(\frac{M_1}{l_1} + \frac{M'}{l_2}\right), \quad . \quad . \quad . \quad . \quad (38)$$

and for equal panels

$$R_2 = W - \frac{M_1 + M'}{l}. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

As previously stated, one or both of the two middle drivers always governs for equal panels, and usually this is true also for unequal panels. The moments M' with reference to these drivers are given in the moment table for Cooper's E–60 Loading. Eqs. (38) and (39) are somewhat more convenient than Eqs. (36) and (37), because the moment M_2 in the latter cannot be read directly from the moment table. If M_2' denotes the moment of W about the last wheel to the right and a the distance from that wheel to R_3,

$$M_2 = M_2' + Wa. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which M_2' and W may be read from the moment table.

If both middle drivers satisfy the criterion, R_2 must be found for each of these positions to determine which is greater. The difference between these two values is usually very slight.

Example. Required the maximum load on a floor beam from Cooper's E–60 Loading for a 28-ft. panel length.

Solution. On reference to the moment table it is seen that if wheel 12 is placed at the floor beam, the last wheel in the panel to the left is wheel 8, and the last to the right, wheel 16. Hence $W = 213$ and $W_1 = 84$ or 114 according to whether the critical load 12 is assumed to act at a differential distance to the right or to the left of the support. Since $213 > 84 \times 2$ and $< 114 \times 2$ the criterion is satisfied.

First Method. Wheel 16 is 4 ft. from the right support. Hence from Eq. (40),

$$M_2 = 5090 + 213 \times 4 = 5942,$$

and by Eq. (37),

$$R_2 = \frac{5942 - 2 \times 1262}{28} = 122.1,$$

which is the desired floor-beam reaction in thousands of pounds.

Second Method. By Eq. (39),

$$R_2 = 213 - \frac{1262 + 1289}{28} = 121.9.$$

It may be shown similarly that wheel 13 also fulfills the criterion. For that position wheels 9 to 7, inclusive, are on the double panel, and $W = 213$ as before.

By Eq. (39)

$$R_2 = 213 - \frac{1227 + 1280}{28} = 123.5,$$

which is slightly greater than the value previously found.

46. Series of Floor-Beam Loads. It is sometimes necessary or desirable, as, for example, in the design of skew bridges, bridges on curves and swing bridges, or of trusses with more than one web system, to determine the loads on a series of consecutive floor beams from a concentrated-load system. As in the preceding article, the stringers spanning the successive panels l_1, l_2, l_3, etc., are assumed to act like simple beams. Let M_1, M_2, etc, (Fig. 36), denote the moments of *all* loads to the left of R_2, R_3, etc., respectively, about these respective supports. For reasons explained in the last article:

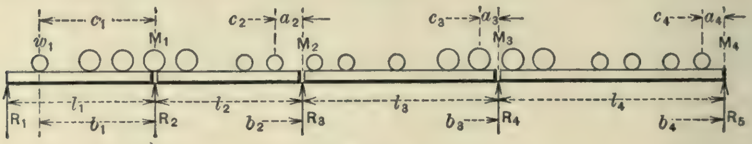


FIG. 36.

$M_1 = R_1 l_1;$

$M_2 = R_1(l_1 + l_2) + R_2 l_2;$

$M_3 = R_1(l_1 + l_2 + l_3) + R_2(l_2 + l_3) + R_3 l_3;$

. . . .

$M_n = R_1(l_1 + l_2 + \ . \ . \ . \ + l_n) + R_2(l_2 + l_3 + . \ . \ . + l_n) + \ . \ . \ . \ + R_n l_n;$

whence

$$R_1 l_1 = M_1;$$
$$(R_1 + R_2) l_2 = M_2 - M_1;$$
$$(R_1 + R_2 + R_3) l_3 = M_3 - M_2;$$
$$. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$
$$(R_1 + R_2 + \ . \ . \ . \ + R_n) l_n = M_n - M_{n-1}.$$

and

$$\left.\begin{aligned}
R_1&=\frac{M_1}{l_1}\\
R_2&=\frac{M_2-M_1}{l_2}-R_1\\
R_3&=\frac{M_3-M_2}{l_3}-(R_1+R_2)\\
&\cdots\cdots\cdots\cdots\cdots\\
R_n&=\frac{M_n-M_{n-1}}{l_n}-(R_1+R_2+\ldots+R_{n-1})
\end{aligned}\right\} \quad . \quad . \quad (41)$$

For the usual case of equal panels, of length l, the following equations may be derived either from Eqs. (41), or directly from the first set of equations for M_1, M_2, etc.

$$\left.\begin{aligned}
R_1&=\frac{M_1}{l}\\
R_2&=\frac{M_2}{l}-2R_1\\
R_3&=\frac{M_3}{l}-(3R_1+2R_2)\\
&\cdots\cdots\cdots\cdots\cdots\\
R_n&=\frac{M_n}{l}-[nR_1+(n-1)R_2+\ldots+2R_{n-1}]
\end{aligned}\right\} \quad . \quad . \quad (42)$$

In Eqs. (41) and (42) each floor-beam load involves the values of all such loads to the left, so that an error in any one of these will affect all succeeding values. This may be avoided by using the following formulas derived from Eqs. (41):

$$\left.\begin{aligned}
R_1&=\frac{M_1}{l_1}\\
R_2&=\frac{M_2-M_1}{l_2}-\frac{M_1}{l_1}\\
R_3&=\frac{M_3-M_2}{l_3}-\frac{M_2-M_1}{l_2}\\
&\cdots\cdots\cdots\cdots\cdots\\
R_n&=\frac{M_n-M_{n-1}}{l_n}-\frac{M_{n-1}-M_{n-2}}{l_{n-1}}
\end{aligned}\right\} \quad . \quad . \quad (43)$$

For equal panels these become:

$$\left.\begin{aligned} R_1&=\frac{M_1}{l}\\ R_2&=\frac{M_2-2M_1}{l}\\ R_3&=\frac{M_3-2M_2+M_1}{l}\\ &\cdots\cdots\cdots\cdots\\ R_n&=\frac{M_n-2M_{n-1}+M_{n-2}}{l} \end{aligned}\right\} \quad . \quad . \quad (44)$$

The values of R_2 in Eqs. (43) and (44) are seen to agree respectively with Eqs. (36) and (37) of the preceding article, as they should.

In applying these formulas the moments M_1, M_2, etc., may be readily found as indicated in Eq. (40), Art. 45. The distance a in that equation becomes for successive panels a_1, a_2, a_3, etc., as seen in Fig. 36. To find these distances, first compute the distances b_1, b_2, b_3, etc., from load w_1 to each successive support; then read from the moment table the distances c_1, c_2, c_3, etc., from w_1 to the farthest load on each span; and finally $a_1=b_1-c_1=0$ (with a wheel at R_2), $a_2=b_2-c_2$, etc.

Problem. A single-track bridge contains eight 28-ft. panels. Assume Cooper's E–60 Loading applied to the bridge with wheel 4 at the right end of the left end panel. Compute the floor-beam loads by Eqs. (42) and (44), with the aid of the moment table. Note the time required by each method.

CHAPTER V

APPLICATION OF THE LAWS OF EQUILIBRIUM TO THE DETERMINATION OF STRESSES

A. GENERAL CONSIDERATIONS

47. A Truss is a framed structure consisting of members so proportioned, and so fastened at their extremities, that the structure is stable under the action of the loads which it is designed to carry.

In the analysis of the stresses in a truss the following assumptions are implied unless otherwise stated:

1. That the effect of the strains in the members on the reactions and the stresses is neglected (Art. 12).
2. That the axes of the members and the lines of action of the external forces lie in a common plane.
3. That the axes of the members meeting at any joint intersect at a common point.
4. That the joints at which the members are united are frictionless hinges.

While the truss as a whole acts like a beam, the individual members of the truss are not subjected to bending from external forces applied at the joints, but the stresses in these members from such loads are either purely tensile or compressive. Unless a member is vertical it is subjected to bending from its own weight. The stresses from this cause are relatively so small, however, that they may usually be neglected without sensible error. Framed structures are generally so designed that extraneous loads are imparted to the trusses only at their joints.

The triangle is the only geometric figure whose form cannot be altered without a change in the length of its sides. Hence a truss formed of a collection of triangles will necessarily be rigid, provided each member is capable of sustaining the stress to which it is subjected. If *m* denotes the number of members, and j the number of joints, then for a triangle, $m=3$ and $j=3$. For each additional joint, two additional members are required, so that in general

$$m=2(j-3)+3=2j-3. \quad (1)$$

If a truss contains fewer members than indicated by this equation it will not be stable under a load applied at any joint selected at random, although it may be stable under some special form of loading. Thus in Fig. 1 (*a*), $j=6$; therefore, by Eq. (1), m should equal 9, whereas the actual number of members is 8. If the truss is symmetrical about a vertical axis, and if two equal loads, W, are applied as shown, the truss will be stable; but if these loads are unequal, or only one of these loads is applied, the truss will collapse by angular movement at its joints. If, however, another member Bc or Cb is added, as in Fig. (*b*), Eq. (1) will be fulfilled, and the truss will then be stable under a load applied at any joint.

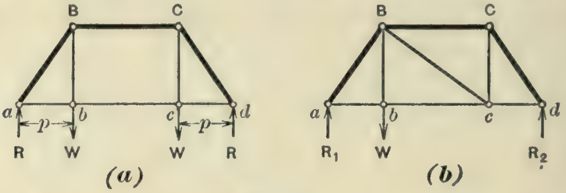


FIG. 1.

The stresses in a truss which satisfies Eq. (1) are statically determinate (Art. 12), provided (1) the requirements for the supports stated in Art. 23 (last paragraph) are met, and (2) the members are properly disposed. Thus if in Fig. 2 member Bc is removed to the position Dc, m and j will be unaffected, but the truss will no longer be stable. Condition (2) means generally, although not necessarily (see Art. 54, Problem), that the truss shall consist of a collection of triangles. Condition (2) is fulfilled when the length of every member is independent of the length of all the other members.

If the number of members in a truss exceeds that required by Eq. (1), the truss contains *redundant* members, and the stresses are then, in general, statically indeterminate. In that case the stresses can be found by the principles of statics, only if the redundant members or their connections are so designed that the stresses in these members are known to be zero for the loading under consideration, and that these members may therefore be regarded as having no existence under that loading. Thus if another diagonal member, *Cb*, is added in Fig. 1 (*b*), the truss becomes redundant. The load *W* at *b* will then tend to produce tension in *Cb* and compression in *Bc*, but the magnitudes of these stresses cannot, in general, be found by the principles of statics. If, however, these members are so flexible that they cannot act in compression, but only in tension, the stress in *Bc* from a load at *b* will be zero; and similarly, for a load at *c*, the stress in *Cb* will be zero. By neglecting the member whose stress is known to be zero, the stresses become statically determinate. If the redundant members are stiff their connections may be so designed that the members cannot act in tension, but only in compression.

Since Eq. (1) serves to determine the existence of redundant members in a truss and is also applicable as a test of stability, it may be called the *criterion for redundancy* or the *criterion for stability*, according to the conditions under consideration.

48. Principal Elements of a Truss. Fig. 2 represents a truss of span l having six equal panels of length p. The points A, B, b, etc., are called 'joints,' 'apexes' or 'panel points'; AG is the 'upper chord' or 'top chord,' bf is the 'lower chord' or 'bottom chord.' The distance h, between the axes of the chords, is called the 'height,' or 'depth,' of the truss.

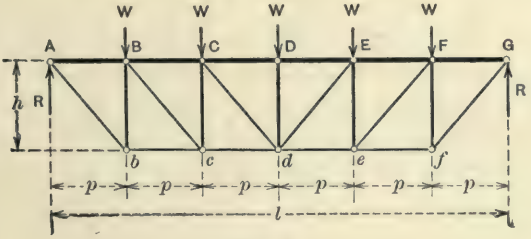
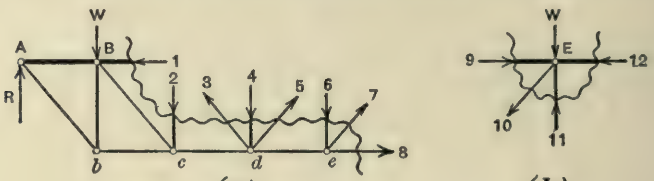


FIG. 2.

The members between the chords are known as 'web members' or 'braces.' Collectively, the web members are called the 'web system' of the truss. The vertical web members are called 'verticals' and the inclined web members, 'diagonals.'

Under the action of the five equal loads W, certain truss members, distinguished by heavy lines in the figure, are in compression, while other members, distinguished by light lines, are in tension. Members which are in compression under every form of loading for which the truss is designed are called 'compression members,' and members whose stress is always tensile are called 'tension members.' Members designed to resist either tension or compression, according to the loading to which the truss is subjected, are said to be 'counterbraced,' or are called 'counterbraces.' Compression web members are also called 'posts,' or, if they are relatively small, 'struts.' Long compression members in buildings or in viaducts are known as 'columns.' Tension web members are also called 'ties.' Truss members having other special functions will be defined later.

49. Laws of Equilibrium Applied to Stresses. The requirements for the equilibrium of any part of a truss are stated in Art. 3. Figs. 3 (*a*) and (*b*) represent parts of the truss shown in Fig. 2, of the preceding article. If in Fig. (*a*) forces 1–8 equal in magnitude and direction to the stresses in the member severed, are applied to those members, these forces, jointly with the original external forces R and W, will evidently hold that segment of the truss in equilibrium. The forces 1–8 may be regarded as external forces with respect to the severed segment of the truss, and these forces, together with R and W, constitute now a set of external forces in equilibrium, which must fulfill the three laws of equilibrium. It follows therefore that if the stresses in three of the members cut are unknown their values may be found from the three conditions of equilibrium, $\Sigma V=0$, $\Sigma H=0$ and $\Sigma M=0$, provided the three stresses are non-concurrent (Art. 18). Hence, if a truss can be divided into two parts by a section cutting only three members, not meeting at a common point, the stresses in those members can



(*a*) (*b*)

FIG. 3.

be found by the principles of statics. If, as in Fig. (*b*), the section is taken around a joint, so that the forces W and 9–12, which hold the segment in equilibrium, are concurrent, then, for reasons explained in Art. 18, the unknown stresses can be found only it they do not exceed two in number.

The truss shown in Fig. 4 cannot be subdivided by any section cutting member CD without severing more than three members. Thus section 1 cuts four members. From this it does not necessarily follow that the stresses in the truss are statically indeterminate. For, by taking the section 2, cutting only three members, stress cd can be found. Taking now section 1, the stresses in only three of the four members cut are unknown, and since these members do not meet at a common point, their stresses can be found. Applying the criterion for redundancy, Eq. (1), Art. 47, to this truss, $j=15$ and $m=27$. Since $27=2(15-3)+3$, the criterion is fulfilled, thus showing that the stresses in this truss are statically determinate.

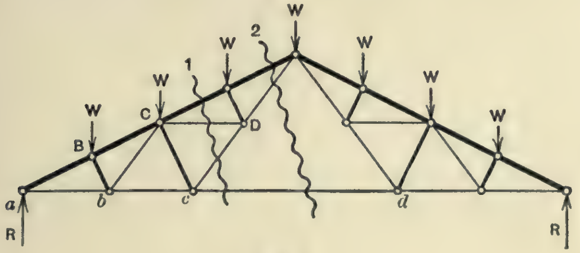


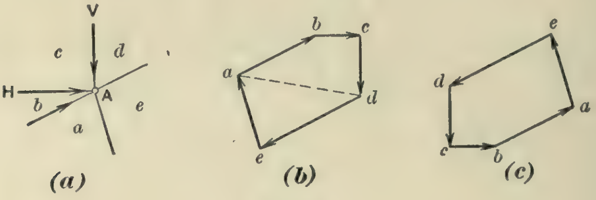
FIG. 4.

The unknown stresses—two for concurrent, three for non-concurrent stresses—may be found either graphically or by computation. The latter method is also known as the 'algebraic' or the 'analytic' method, although in its application the processes are usually wholly numeric. The various methods of determining stresses, and the conditions under which a given method may be applied to advantage, will now be considered.

B. CONCURRENT STRESSES

50. Concurrent Stresses. Two Stresses Unknown in Magnitude (Including Direction). (*a*) *Graphic Method.* In Fig. 5(*a*) let *H* and *V* denote the external forces (loads) at any joint, *A*, and let *de*, *ea* and *ab* denote the truss members which intersect at that joint. Assume that stress *ab* is fully known, and that stresses *de* and *ea* are known as to position, but unknown as to magnitude (including direction).

Since the five forces are in equilibrium, they must be capable of being formed into a closed force polygon. Beginning at some point, *a*, Fig. (*b*), lay off the known forces *ab*, *bc*, and *cd* to a suitable scale, the order being determined by proceeding around joint *A* in a clockwise direction. From *d*, draw *de*, and from *a*, draw *ae*, parallel to members *de* and *ea* respectively, and in such directions as to ensure the intersection of these lines at some point, *e*. The lengths of the lines *de* and *ea* represent then the magnitudes of the stresses in the members to which they are respectively parallel, and which are similarly lettered. For equilibrium the forces must point around the polygon in the same direction—i.e., clockwise—as shown by the arrows. If arrow *de* is assumed to be applied at the free end of member *de*, it will point *toward* the joint, thus indicating that member *de* is in *compression*. It is seen, similarly, that member *ea* is in compression. The same results might have been obtained by laying off the forces in a counter-clockwise direction, as in Fig. (*c*). In the construction of force polygons representing forces meeting at a common point, a clockwise procedure around that point is, however, more natural, and is followed universally in practice.



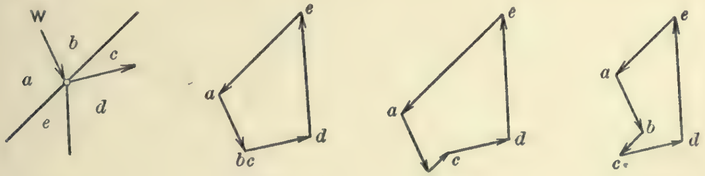
(*a*) (*b*) (*c*)

FIG. 5.

The broken line *da* in Fig. (*b*) represents the equilibrant of the three given forces in a direction from *d* to *a*. The two required forces *de* and *ea* are seen to be components of this equilibrant, as they should be. The problem in this case is thus seen to be merely a variation of that in Art. 18 (Fig. 7), in that instead of finding the equilibrant of certain given forces, it is required now to find the components of this equilibrant along two given lines of action, corresponding to the given positions of two truss members whose stresses are unknown.

(*b*) *Algebraic Method.* To find the two unknown stresses algebraically, or by numeric computation, usually the most convenient method is to assume two axes coincident with the lines of action of the two unknown stresses. From the given magnitudes and inclinations of the known forces, these forces may each be resolved into components in the directions of the assumed axes. Each of the required stresses is then equal in magnitude and opposite in direction to the resultant (algebraic sum) of the components along the axis coincident with the line of action of that stress. This method of finding stresses is known as the method by *resolution of forces.*

51. Common Special Cases in Concurrent Stresses. *Case I.* If the two members whose stresses are unknown lie in the same straight line the magnitudes of these stresses cannot be determined from an analysis of the set of concurrent forces to which they belong. The resultant, or the algebraic sum of these two stresses, and the magnitude of the stress in a third member, can, however, be found.



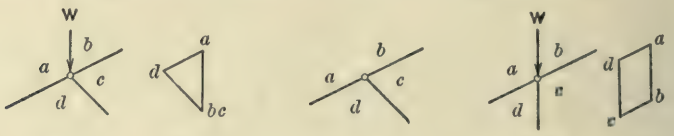
(*a*) (*b*) (*c*) (*d*)

FIG. 6.

Thus in Fig. 6(*a*), let the stresses in members *bc* and *ea*, lying in the same straight line, and the stress in a third member, *de*, be unknown. Begin the force polygon, Fig. (*b*), at *a*, and lay off the known forces *ab* and *cd*. From *d*, draw *de*, and from a draw ae, parallel to members de and ea respectively. Then de represents the magnitude of the stress in member de. The force de acts from d to e in the force polygon, thus indicating compression. However, ea does not denote the stress in member ea, but it represents the algebraic sum of the unknown stresses ea and bc. The force polygon may be constructed as in Figs. (*c*) or (*d*), since bc may be drawn in either direction from b, and without limit as to length. The data given do not suffice, therefore, to determine the stresses bc and ea. The algebraic sums of forces bc and ea in Figs. (*c*) and (*d*), respectively, are equal to force ea in Fig. (*b*). The forces de in these three figures have, of course, the same magnitude.

The stress in member de may also be found algebraically by the method described in the preceding article.

Case II. Fig. 7 represents the common case in which three members, of which two lie in the same straight line, meet at a joint. If the load W is the only known force in Fig. (*a*), the force polygon may be drawn as in Fig. (*b*), in which cd denotes the stress in member cd, and da the algebraic sum of stresses da and bc whose magnitudes are, however, indeterminate.



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 7.

If no load exists at the joint, as in Fig. (*c*), stress cd is zero, and stresses da and bc are equal. In the analysis of the stresses in a truss such members as cd may therefore be disregarded, if the other two members lie in the same straight line and no load occurs at the joint.

If the load W and the third member cd also lie in the same straight line, as in Fig. (*d*), stress $cd = W$, and stress $bc =$ stress da. The force polygon is of the form shown in Fig. (*e*), although, if the load W is the only known force, the lengths of the parallel sides bc and da are obviously indeterminate. It is evident also that if ab represents a truss member instead of a load, stress $ab =$ stress cd and stress $bc =$ stress da as before.

Case III. Another common case is shown in Fig. 8, in which two horizontal chord members and two web members, of which one is vertical, meet at a joint which is also subjected to a vertical load, W. Suppose that W, H_1 and H_2 are known and that $H_1 < H_2$. Then in Fig. (*b*), beginning at *e*, lay off the known forces *ea*, *ab*, and *bc* as shown by the full lines. Completing the force polygon, $cd = D$ (tension) and $de = P$ (compression). If $H_1 > H_2$, stress D is compressive and P tensile, as shown by the broken lines in Fig. (*b*).

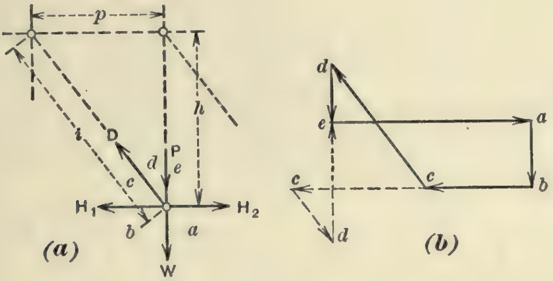


FIG. 8.

As will be seen later it is usually preferable to find the unknown stresses in such a case by the method of resolution of forces explained in Art. 50 (*b*). Since the forces are, in this case with one exception, vertical and horizontal, it is not advantageous to resolve the known forces along axes coincident with the unknown forces D and P. From $\Sigma H = 0$, $H_2 - H_1 =$ horiz. comp. of D. Therefore from the known dimensions of the truss the stress in the diagonal is

$$D = (H_2 - H_1)\frac{i}{p}. \qquad \ldots \ldots \ldots \quad (2)$$

Again from $\Sigma V = 0$, $P + W =$ vert. comp. of D. Hence the stress in the post is

$$P = D\frac{h}{i} - W = (H_2 - H_1)\frac{h}{p} - W. \qquad \ldots \ldots \quad (3)$$

A thorough understanding of the three simple cases considered in this article will be found indispensable in connection with the general analysis of stresses in trusses.

52. Stresses in Trusses by Graphic Method. The stresses in a truss may be found graphically by drawing a force polygon for the forces at each joint in turn, provided not more than two forces at each joint are unknown. Instead of drawing a separate polygon for each joint, however, the several polygons may be conveniently combined in a single *stress diagram* as will now be shown. The stress diagram is composed of the force polygons for (1) the external forces and (2) the forces (loads and stresses) meeting at each joint. Such diagrams are known as Maxwell diagrams.

Let it be required to find the stresses in the truss shown in Fig. 9(*a*), from the five equal loads *W*. The reactions, *R*, equal 2.5 *W*. Proceeding in a clockwise direction, supply the letters *A*, *B*, *C* . . . *O*. In Fig. (*b*) lay off the force polygon of the external forces, i.e., *ab*, *bc*, *cd*, *de*, *ef*, *fg* and *ga*, thus closing the polygon at *a*. The force polygon is in this case reduced to a single line called a *load line*. At joint 1, *R* is known, and the stresses in *AH* and *HG* are unknown. Beginning at *g*, Fig. (*b*), the polygon for this joint is *gahg*. The forces must point in the same direction around this polygon. Since *ga* acts from *g* to *a*, *ah* must act from *a* to *h* and *hg* from *h* to *g*. If arrows pointing in these directions are assumed to be placed upon the members in question, the arrow on *AH* will point toward joint 1, and that on *HG* will point away from that joint. Member *AH* is therefore in compression (−) and member *HG* is in tension (+).

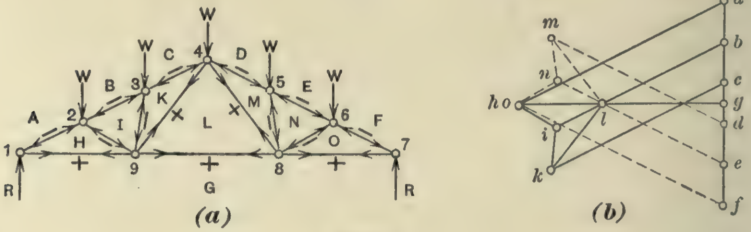


FIG. 9.

Proceeding now to joint 2, the stresses in *BI* and *IH* are unknown. The known forces are the stress in *HA*, which is compressive, and therefore acts toward joint 2, and *W*. In entering the force polygon, begin at *h*, since *ha* acts, with respect to joint 2, from *h* to *a* and not from *a* to *h*. The polygon for joint 2 is *habih*, the forces all pointing around the polygon in the direction in which these letters succeed each other. The stresses in *BI* and *IH* are therefore compressive (−). For joint 3, begin at *i*. The polygon is *ibcki*. At joint 9, all forces except the stresses in *KL* and *LG* are now known. Beginning at *g*, the polygon is *ghiklg*.

The stresses in all members in the left half of the truss have now been found. Those in the right half follow by symmetry of truss and loading. They are shown by the broken lines in Fig. (*b*), although these lines need not be drawn.

It is important to observe that *the forces at each joint must be considered in* ***successive*** *order and in a* ***clockwise*** *direction around that joint*, since the sequence of the letters *A*, *B*, *C*, etc., in Fig. (*a*) was chosen in a clockwise direction around that figure. With respect to joint 1, the stress in member 1–2 is therefore *ah*, and not *ha*. In the stress diagram the direction *ah*, i.e., from *a* to *h*, indicates that the stress in *AH* acts toward joint 1, and that the member 1–2 is in compression. On the other hand, proceeding in a clockwise direction around joint 2, the stress in member 1–2 with respect to that joint is *ha*, and not *ah*. In the stress diagram the direction *ha*, i.e., from *h* to *a*, indicates that stress *ha* acts toward joint 2, and that the member 1–2 is in compression. If a member is in *compression* its stress is exerted *toward* the joints at both extremities of the member; if a member is in *tension* its stress is exerted *away from* the joints at both extremities of the member. The directions in which the stresses are exerted toward each joint are shown by the arrow-heads in Fig. (*a*). With the above explanation it is not necessary that such arrow-heads should be shown in similar figures hereafter.

Practical Hints. To ensure accurate results by the graphic method, it is especially important that the truss diagram should be drawn very carefully and to a liberal scale. After the force polygon for any given joint has been drawn, the character of the stress in each of the members meeting at that joint should be observed, and the appropriate sign, + or −, should be noted at once on the truss diagram. On completion of the stress diagram, the magnitude of all the stresses should be scaled, and entered on the corresponding members in the truss diagram. It is convenient to express the stresses in terms of thousands of pounds. The diagrams should be drawn with a hard pencil, sharpened to a chisel edge. In practice, such diagrams are seldom 'inked in'; but if this is to be done, the stresses should be scaled before inking. For very accurate work it is desirable to indicate the points of intersection in both diagrams by pin pricks, surrounded by little circles in which the lines terminate.

53. Inclined Loads. Let the roof truss, Fig. 10(*a*) be assumed to rest on rollers at the left support and to be fixed at the right support. Required, the stresses from the wind loads shown in the figure on the left half of the truss, normal to the upper chord.

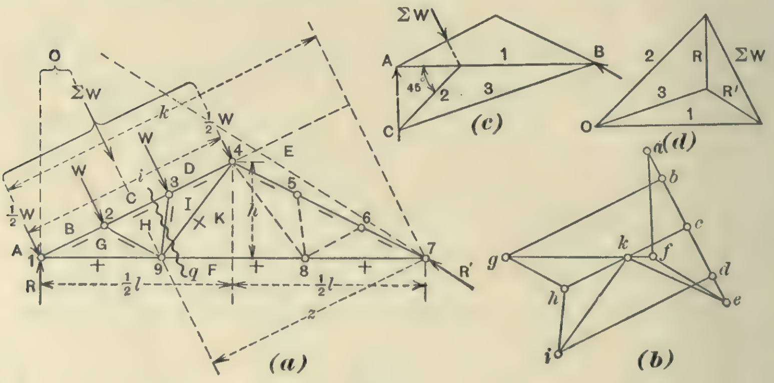


FIG. 10.

Reactions. Graphic Methods. The resultant, ΣW, of the loads occupies the central position shown. The position of R' is found by producing the line O–7, O being the intersection of R and ΣW (Art. 25). The magnitudes of the reactions are found from *ef* and *fa* Fig. (*b*) in the force polygon, *abcdefa*, of the external forces.

If the truss diagram is drawn to a large scale the point O may be off the paper. The reactions may then be found graphically by the following simple construction.

In Fig. (*c*), which represents the truss in outline, produce ΣW to its intersection with AB. Draw 2 at an angle of 45° (for convenience) with AB to its intersection C with a vertical through A, and draw CB, thus closing the equilibrium polygon 1–2–3. In Fig. (*d*) lay off ΣW and locate the pole O by the rays 1 and 2 drawn parallel to the corresponding segments of the equilibrium polygon.

Then find the reactions by drawing ray 3 parallel to the closing line 3.

Algebraic Methods. By moments about 7, $R = \Sigma W \times z/l$. To find z: From similar triangles, $k/l = \frac{1}{2}l/i$. Hence $k = l^2/2i$, and $z = k - \frac{1}{2}i = l^2/2i - \frac{1}{2}i$.

If R_h' and R_v' denote respectively the horizontal and vertical components of R', $R_h' = \Sigma W \times h/i$ and $R_v' = \Sigma W \times \frac{1}{2}l/i - R$. Finally, $R' = [(R_v')^2 + (R_h')^2]^{\frac{1}{2}}$.

The reactions may also be found by combining the reactions from the vertical and horizontal components, V and H, of ΣW. If these components are assumed to act at the intersection of ΣW with the rafter,

$$R = \tfrac{3}{4}V - \frac{H \times \frac{1}{2}h}{l},$$

and

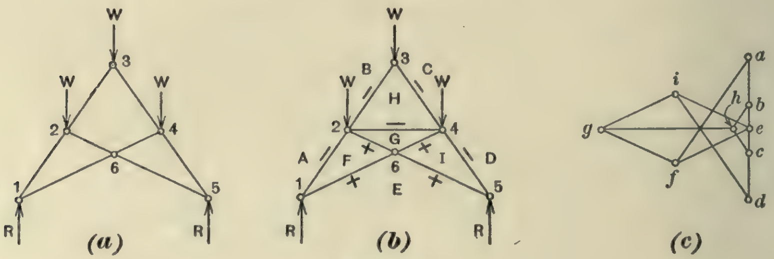
$$R' = [(V - R)^2 + H^2]^{\frac{1}{2}},$$

in which $V = \Sigma W/i \times \frac{1}{2}l$ and $H = \Sigma W/i \times h$.

If the components V and H are assumed to act at the intersection of ΣW with the lower chord, and if u denotes the distance 7–9 of this point from the right support, $R = Vu/l$ and R' may be found as before.

Stresses. Referring to joints 5, 6, and 8, it is seen that the stresses in members 5–8, 6–8, and 4–8 are zero (Art. 51, Fig. 7*c*). These members, shown by broken lines, may therefore be disregarded. The letters *A*, *B*, *C*, etc., may now be supplied, and the force polygons for joints 1, 2, 3, 9, and 4, in the order stated, may be drawn in Fig. (*b*) as explained in the preceding article. On reaching the last joint, 4, the stress in *EK* (member 4–5) is the only unknown. The points *e* and *k* in the force diagram are already located. Hence a line drawn through *e*, parallel to *EK*, should intersect *k*. This serves as a check on the accuracy of the work. If the line does not intersect *k*, the perpendicular distance from *k* to this line is called the 'error of closure.' The line *ek* is called the 'closing line' in the stress diagram. Unless the error of closure is very small, the stress diagram should be redrawn. Inaccuracies in the truss diagram will not affect the closing of the stress diagram, provided the loads are correctly located with respect to the supports, so that the inaccuracies do not affect the reactions. However, if the truss diagram is drawn inaccurately, the resulting stresses are obviously inaccurate. The magnitude of the error of closure is not a direct criterion of the accuracy of the stresses, inasmuch as inaccuracies in the stress diagram are apt to be of a compensating nature.

If the truss is divided into two parts by any section as, for example, q, the external forces to either side of that section together with the stresses in the members cut, constitute a system of forces in equilibrium, and these forces must therefore form a closed polygon in Fig. (*b*). Thus the forces to the left of section q are represented by the polygon, *abchikfa*.



(*a*) (*b*) (*c*)

FIG. 11.

54. Scissors Truss. The frame shown in Fig. 11 (*a*) is stable only if both reactions are inclined. It is, in fact, not a truss, but a three-hinged arch. This becomes immediately apparent if members 2–3 and 3–4 are conceived to be removed. To prevent rotation of the triangles 1–2–6 and 4–5–6 about the central hinge 6, horizontal reactions must evidently be added at the supports. It is clear that this rotation cannot be prevented by restoring the members 2–3 and 3–4, hinged at 3. Again, applying the criterion for stability (Eq. (1), Art. 47), $j=6$; therefore m should equal $2(6-3)+3=9$, whereas the actual number of members is 8. Stability, with vertical reactions only, may be ensured by introducing the additional member 2–4, Fig. (*b*), or by an added member from 3 to 6 or from 1 to 5.

In drawing the force polygons in the stress diagram, Fig. (*c*), for successive joints, beginning with joint 1, it will be found on passing to joint 2 that the stresses in three members (*BH*, *HG*, and *GF*) are unknown. Hence the force polygon for this joint cannot be drawn. Passing to joint 3, however, only two stresses are unknown, so that the polygon *bchb* can be constructed. Since the stress in *BH* is now known, there remain only two unknowns at joint 2, so that the polygon *fabhgf* may now be drawn. Note that *ef*=*gi* and *ie*=*fg* (Art. 51, Fig. 7 *d*).

Problem. Supply member 1–5 in Fig. 11 (*a*), and find the stresses for $W = 3000$ lbs. Compare these stresses with those for like loading in the truss shown in Fig. (*b*). Observe that with member 1–5 supplied the truss is stable, although it does not consist of a collection of triangles, since it contains the quadrilateral 2–3–4–6 (Art. 47).

55. Fink Truss. A common form of roof truss is that shown in Fig. 12 (*a*). Since the truss as well as the assumed loading is symmetrical the stresses in only the left half of the truss will be found. The lower half of the load line, Fig. (*b*), is therefore omitted. Bisect *ef* at *g*, then *ga*=*R*. Construct the force polygons for joints 1, 2, and 8, in the order named. Passing then either to joint 3 or 7, there will be three unknowns, so that the polygons for these joints cannot be drawn. At joints 4, 5, and 6, there are also three or more unknowns. The problem may be solved as follows:

First General Method. Assume the members 3–6 and 4–6 removed, and a new member, 4–7, inserted instead. The transformed truss will be stable, and the stresses in *EN*, *NO* and *OG* will be unaffected; for taking section *q*, the three stresses named must hold the external forces to either side in equilibrium, as before. The polygons for joints 3 and 4 in the transformed truss may now be constructed, the former as *kicdzk* and the latter as *zdenz*. The stress *en*, thus found, is the correct stress in member *EN* of the original truss. If the auxiliary member 4–7 is now removed, and the original members 3–6 and 4–6 are restored, there are only two unknowns at joint 4, and the polygon *denmd* for that joint may be completed. Passing then to joint 3, the stresses in *ML* and *LK* are the only unknowns. The polygon for that joint is *icdmlki*.

Special Method. If the loads at joints 2 and 4 are equal and parallel, as in this case, then on resolving these loads into components normal to and coincident with the upper chord, it is seen that the former represent the stresses in *HI* and *MN*, and that these stresses are equal. By symmetry, the stresses in *IK* and *LM* are then also equal. This may be seen definitely by resolving the forces meeting at 6 along the axes 3–6 and 5–6. The stress in *LM* equals the component of the stress in *MN* along axis 3–6. Similarly, at joint 8, the stress in *IK* equals the component of the stress in *HI* along axis 3–8, which makes the same angle with axis 1–8 as 3–6 with 5–6. Since members *IK* and *LM* make equal angles with *KL*, it follows that under the conditions assumed *lm*, Fig. (*b*), drawn from any point *l* on *kl*, parallel to *LM*, must be equal to *ki* and make the same angle with *kl*. Hence *hi* and *mn*, Fig. (*b*), must lie in the same straight line. After *h* has been located, these lines may therefore be drawn at right angles to *bh*, thus determining the stresses *ci*, *dm*, and *en* in the upper chord members. At joint 3 there are then

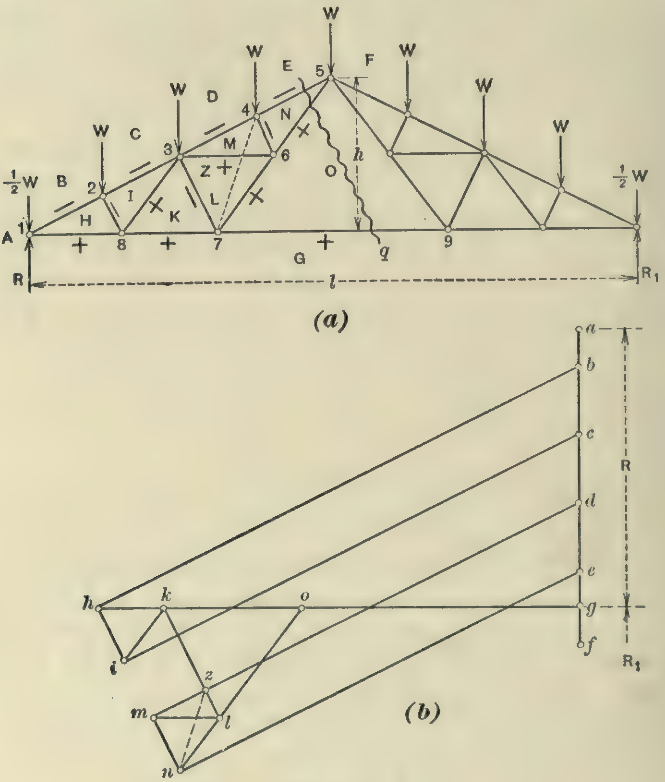


FIG. 12.

only two unknowns, so that the polygon for that joint may be drawn without introducing the imaginary member 4–7. This method is applicable only under the special conditions of equal and parallel loads at 2 and 4—conditions which are, however, usually fulfilled.

Second General Method. After the polygons for points 1, 2, and 8 have been drawn, the polygon for joint 7, at which there are three unknowns, may be constructed if the stress in OG, one of the unknowns, has been found by computation. Taking section q, the stresses in the three members cut, jointly with the external forces to the left of the section, must be in equilibrium. From $\Sigma M=0$, taking moments about 5, and letting M_5 denote the resultant moment of the forces to the left of the section about 5, stress $og=M_5/h$. Laying off this stress from g to o in Fig. (*b*), the force polygon for joint 7, $ogklo$, may now be completed, and passing to joint 3, the stresses in DM and ML are now the only unknowns, so that the polygon for this joint may be drawn in the usual manner.

In computing M_5, assume one-half of the load W at 5 to act immediately to the left of that point. Then the resultant of the loads to the left of 5 acts at 3 and is equal to the reaction R. Hence $M_5=R\times\frac{1}{2}l-R\times\frac{1}{4}l=\frac{1}{4}Rl$, and stress $og=\frac{1}{4}Rl\div h$. The rise, h, of an ordinary roof truss is commonly $\frac{1}{4}l$, in which case stress $og=R$.

Obviously this method is not strictly graphic. The moment M_5, which is seen to be the bending moment at a vertical section through 5, may be found graphically by the method explained in Art. 37 (*b*), but it is evidently much simpler to find this moment as above by computation.

It is sometimes necessary to ascertain, or to check, the character of the stress in a particular member of a truss, from the general stress diagram. In that case it is only necessary to determine the direction in which this stress acts by entering any force polygon in which this stress occurs, beginning with a force whose direction is known, and continuing in the same direction around the polygon until the stress whose character is required has been reached. Thus, in Fig. 12, let it be required to find the character of the stress in NO. Taking section q, the stresses in the three members cut and the external forces to the left are known to be in equilibrium; the direction of *DE* is also known. Therefore in Fig. (*b*) begin at *d* and proceed in the direction *de* along the path *deno.* Thus *no* is seen to act in the direction *no*, i.e., from *n* to *o*. Hence the stress in *NO* acts away from 6, and is therefore tensile. Again, considering joint 6, stress *MN* is seen to be compressive by inspection of joint 4. Therefore beginning at *m*. Fig. (*b*), proceed in the direction from *m* to *n*, and from *n* to *o*, whence *no* is found to act in the same direction as before.

56. Crane Truss. The force polygon of the forces meeting at a given joint is sometimes represented by a polygon whose sides are the truss members themselves. The stresses may then be readily computed from the known lengths of these members, and the known external forces.

Let it be assumed that the members of the crane truss, Fig. 13, all lie in a common plane, and that the load *W* acts vertically, i.e., parallel to the 'mast' *BC*. The reactions may be most conveniently found by computation. Thus, $R = Wb/a$ and $R_1 = R + W$. Triangle *ABC* is a force polygon of the forces meeting at *A*. Therefore to the same scale to which *BC* represents *R*, *AC* and *AB* represent the stresses in those members. The character of these stresses may be found as usual by proceeding around the triangle in the direction *CBAC*, or by inspection of the requirements for equilibrium at joint *A*. Thus

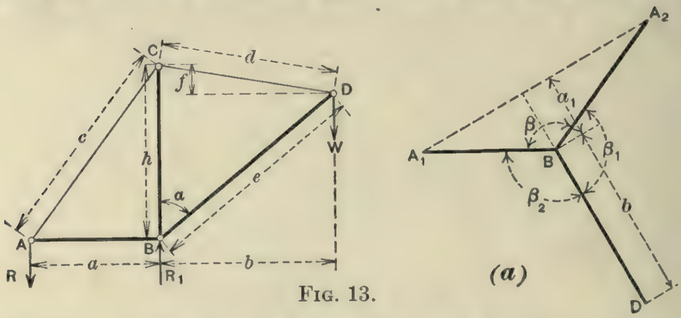


FIG. 13.

$$\text{Stress } AC = +\frac{R}{h}c, \quad . \quad . \quad (4) \qquad \text{and} \qquad \text{Stress } AB = -\frac{R}{h}a. \quad . \quad (5)$$

Again, triangle *BCD* is a force polygon of the forces meeting at *D*. Hence

$$\text{Stress } BD = -\frac{W}{h}e, \quad . \quad (6) \qquad \text{and} \qquad \text{Stress } CD = +\frac{W}{h}d. \quad . \quad (7)$$

Considering joint C, from $\Sigma V=0$, stress BC is equal to the sum of the vertical components of stresses AC and CD, i.e.,

$$\text{Stress } BC=-\left[R+(\text{stress } CD)\frac{f}{d}\right]=-\left(R+\frac{W}{h}f\right) \quad . \quad (8)$$

If the angle α is varied, CD is the only member whose length changes. By Eq. (6), the stress in the 'boom' or 'jib' BD is therefore constant, irrespective of variations in α, and, by Eq. (7), the stress in the 'tie' CD varies directly with d. As α is varied, b changes. The reaction R varies directly with b and, by Eqs. (4) and (5), the stresses in the 'back-stay' AC and the strut AB are affected at a like rate. Again, as α increases up to 90°, both R and f in Eq. (8) increase, and therefore the stress in the mast BC increases.

For values of α greater than 90° the position of the boom for maximum stress in the mast may be found as follows:

Let $\phi=\alpha-90°$. Then $b=e\cos\phi$, whence $R=We\cos\phi/a$, and $f=h+e\sin\phi$. Substituting these values of R and f in Eq. (8), and letting S denote the stress in the mast BC,

$$S=-\left[\frac{W}{a}e\cos\phi+\frac{W}{h}(h+e\sin\phi.)\right]$$

For maximum stress, let $dS/d\phi=0$, whence

$$\frac{dS}{d\phi}=\frac{We}{a}\sin\phi-\frac{We}{h}\cos\phi=0,$$

or

$$\tan\phi=\frac{a}{h}.$$

The ratio a/h is the tangent of the angle between the back-stay and the mast, whence it is seen that the stress in the mast becomes maximum when the boom is swung down to a position perpendicular to the back-stay.

It has been assumed that the members all lie in a common plane, although in reality there are two or more back-stays which lie in different vertical planes. A 'stiff-leg' crane with two back-stays, each capable of acting in compression as well as tension, is shown in plan in Fig. (a). The stresses in the horizontal strut A_1B and in the corresponding back-stay are maximum when the reaction R' at A_1 is maximum. R' may be found by taking moments of the external forces about A_2B as an axis, thereby eliminating the reactions at A_2 and B. Hence

$$R' = Wb \sin \beta_1 / \overline{A_1B} \sin \beta.$$

For maximum R', b and $\sin \beta_1$ must be maximum, whence $b = e$ (boom horizontal) and $\beta_1 = 90°$, and $R'_{\max} = We/\overline{A_1B} \sin \beta$. Stress A_1B is then obtained as in Eq. (5), whence

$$\overline{A_1B}_{\max} = R'_{\max} \times \overline{A_1B}/h = We/h \sin \beta.$$

Similarly, stress A_2B becomes maximum when $\beta_2 = 90°$ and $\overline{A_2B}_{\max} = \overline{A_1B}_{\max}$. The maximum stresses in the back-stays can now be found from their horizontal components, which are the stresses $\overline{A_1B}_{\max} = \overline{A_2B}_{\max} = We/h \sin \beta$.

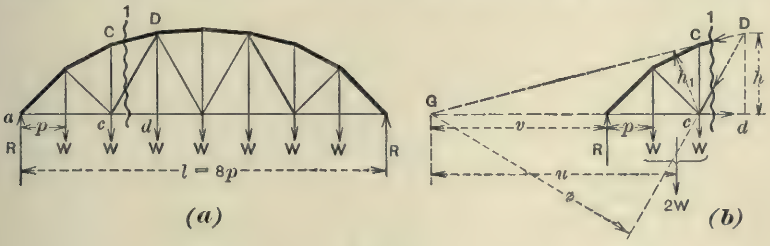
The stress in the mast becomes maximum, however, when the mast is rotated horizontally to that position for which the sum, ΣR, of the vertical components of the stresses in the back-stays becomes maximum. The sum, ΣR, of the reactions at A_1 and A_2 becomes maximum when BD lies in a plane perpendicular to the line A_1A_2. Then $\Sigma R = Wb/a_1$. Usually $A_1B = A_2B$, in which case BD produced will bisect the angle β. For maximum stress in the mast, the inclination of the boom BD must now be such that $\tan \phi = a_1/h$; i.e., the boom must be perpendicular to an imaginary back-stay whose horizontal projection is a_1. If more than two back-stays are stressed simultaneously, the stress in the mast is statically indeterminate.

In the above analysis of stresses the tie CD is assumed to terminate at C. In an actual crane hoisting ropes (usually wire cables) running through 'pulley blocks' near B, C, and D, serve to control the angle α and the vertical movement of W. These ropes are carried down the mast, or in part down the boom, according to the character of the rigging. The tension in these ropes produces compression of the same magnitude in the members between the points at which the pulley blocks are attached, which must be added to the stresses by Eqs. (6) and (8). The stresses in these ropes are some fractional parts of the stress in the tie CD, and of the weight W, depending on the number of 'sheaves' in the pulley blocks. In designing the mast and boom, due allowance must also be made for bending from eccentric (non-axial) stresses due to the eccentric positions of the pulley blocks.

C. NON-CONCURRENT STRESSES

57. Method of Moments. It has been seen (Art. 49) that if a truss can be divided into two parts by a section cutting not more than three members, not meeting at a common point, whose stresses are unknown, the stresses in those members can be found by the application of the three laws of equilibrium.

General Case. If no two of the three unknown stresses are parallel, the stresses may be conveniently found by the *method of moments*, which is merely the application of the third law of equilibrium, $\Sigma M = 0$. Thus in Fig. 14, let it be required to find the stresses in the three members, CD, Dc, and cd, cut by section 1. These stresses, together with the external forces acting on the segment of the truss to either side of the section, must hold that segment in equilibrium. For simplicity consider that segment for which the external forces are fewer, namely the segment to the left of the section, Fig. (*b*). *To find the stress in any member take moments about the intersection of the other two members*, since the moments of the latter about such a moment center are zero. Thus for stress cd, choose D as the moment center, whence



(*a*) (*b*)

FIG. 14.

$$\text{Stress } cd = \frac{R \times 3p - W(p+2p)}{h}.$$

Since $R = 3.5W$,

$$\text{Stress } cd = \frac{7.5Wp}{h}. \qquad \dots\dots \quad (9)$$

Since $R \times 3p > W \times 3p$, the resultant moment of the external forces about D is clockwise. For equilibrium the moment of stress cd about this point must be counter-clockwise. The arrowhead on this member in Fig. (b) must therefore point in the direction shown, indicating that the stress cd is tensile.

For stress CD, choose the moment center c. Then

$$\text{Stress } CD = \frac{R \times 2p - Wp}{h_1},$$

and substituting for R its value,

$$\text{Stress } CD = \frac{6Wp}{h_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which h_1 is the length of a perpendicular from c to CD produced. Since the moment $R \times 2p - Wp$ acts clockwise with respect to c, stress CD must act in the direction of the arrow, and this stress is therefore compressive.

Similarly for stress Dc, choose G, the intersection of CD and cd, as the moment center. Then

$$\text{Stress } Dc = \frac{Rv - 2Wu}{z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

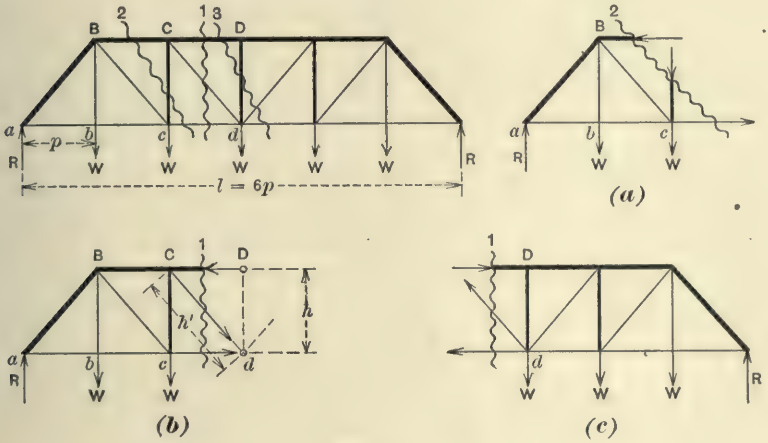
If $Rv > 2Wu$, the resultant moment is counter-clockwise In order that it may be balanced by a clockwise moment the arrow on Dc must point toward c, indicating that stress Dc is compressive. The lever arms u, v and z may be conveniently expressed in terms of the known dimensions of the truss, as will be seen later (Art. 131).

Special Case. If two of the three unknown stresses are parallel, the values of the two parallel stresses may be conveniently found by the method just explained. Thus, in Fig. 14 (b), if h_1 is made equal to h, DC will be parallel to dc. This change will not affect Eq. (9) and in Eq. (10) it is only necessary to substitute h for h_1. The stress in the third member, Dc, may in this case be found by the method described in the next article.

58. Method of Shears. If two of the members cut by a section which divides the truss into two parts, are parallel, the stress in the third member may be conveniently found by the *method of shears*. The parallel members are always chord members, and the third member is a web member.

Special Case. The usual case, in which the chords are horizontal and the loads vertical, is shown in Fig. 15. Taking the section 1, the chord stresses CD and cd may be found by the method of moments explained in the preceding article, the moment centers being chosen at d and C respectively. From Fig. (b) and $\Sigma V=0$, it is seen that the vertical component, S_v, of stress Cd must be equal in magnitude, and opposite in direction to the resultant $R-2W$, of the external forces to the left. Since $R=2.5W$, this resultant, $0.5W$, acts upward. Hence S_v must act downward, indicating that stress Cd must be exerted in a direction away from C, and that it is therefore tensile. Thus

$$\text{Stress } Cd=S_v\,h'/h=0.5Wh'/h.$$



(a)

(b)

(c)

FIG. 15.

Since $0.5W$ is the shear in panel cd, it is seen that *the stress in any diagonal is the product of the shear in the corresponding panel by the secant of the angle between the diagonal and a vertical.* This secant may be most conveniently expressed by the ratio h'/h of the length of the diagonal to the depth of the truss.

Considering the forces to the *right* of the section, Fig. (c) the resultant, $3W-R$ ($=0.5W$), of the external forces now acts downward. Hence S_v must act upward with respect to this segment of the truss. Stress Cd must therefore be exerted in a direction away from d, and is thus found to be tensile, as before.

Taking section 2, the segment to the left of the section is shown in Fig. (*a*). For equilibrium it is seen that stress Cc must be compressive, and equal to the shear in the adjacent panel cd, i.e., stress $Cc = -0.5W$. From $\Sigma V = 0$, applied to the forces meeting at C, Fig. (*b*), it is also seen that stress Cc must be equal to the vertical component, $0.5W$, of stress Cd. From $\Sigma H = 0$, it is evident, Fig. (*a*), that stress BC = stress cd.

Section 3, cutting the next vertical member, Dd, intersects four members, but the stress in this member, from the loading shown, is zero. This is evident from $\Sigma V = 0$, applied to the forces at D. The stress in Dd is equal to the vertical component of any external load at D. If such a load exists, the stress in Dd is found as stated, and section 3 will then cut only three members whose stresses are unknown. These stresses may then be found as above explained.

General Case. In case the lines of action of the external forces are not perpendicular to the parallel chord members cut by a given section, the forces to the left or right of the section may each be resolved into components normal and parallel to the chords. The algebraic sum of the normal components is then the shear in the panel through which the section was taken, and the stress in the diagonal is, as before, the product of this shear by the secant of the angle between the diagonal and a line normal to the chords. If the inclinations of the forces on both sides of the section are irregular, the resultant of the forces to either side may be conveniently found graphically. This resultant may finally be resolved into components normal and parallel to the chords, the former being the desired shear.

Thus Fig. 16 (*a*) represents a segment of an arch. The external forces to the left of section 1 are $W_1 - W_5$ and R. From the force polygon, Fig. (*c*), the resultant of these forces is 5. Resolve this resultant into the components OA and OB, normal and parallel respectively to the parallel members FG and fg, Fig. (*b*). Then stress $Fg = OA \times h'/h$. In order that the component of this stress, in a direction normal to the chords, may balance OA, stress Fg must act in a direction away from F and is therefore tensile.

Stresses FG and fg may be found by moments about g and F respectively. Instead of finding the algebraic sum of the moments of the individual forces, the moment of their resultant may be obtained. To locate this resultant, 5, construct the resultant polygon, 1–5, shown in Fig. (*a*), by drawing its segments parallel to the corresponding rays in Fig. (*c*). Let z and z_1 (not shown) denote the lever arms of 5, Fig. (*b*), about g and F respectively. Then stress $FG = (5) \times z/h$ (compression) and stress $fg = (5) \times z_1/h$ (tension).

The effect of inaccuracies in scaling may be minimized by finding the moments of the resultant, 5, about g and F by multiplying the vertical distances, v and v_1, from g and F, respectively, to 5, by the horizontal projection, H, Fig. (*c*), of the resultant, as explained in Art. 37 (Third method). Thus

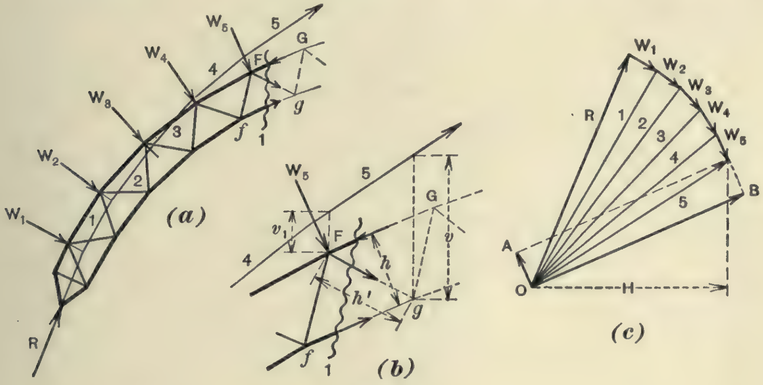


FIG. 16.

$$\text{Stress } FG = H \times v/h$$

and

$$\text{Stress } fg = H \times v_1/h.$$

59. Method of Coefficients. The stresses from uniform vertical loading, in a truss with horizontal chords, may be conveniently found by the *method of coefficients*. This method derives its name from the fact that the stress in each member is expressed as the product of a constant by a numeric ‘coefficient.’ In the common case of a truss with equal panels and vertical posts, to which this method is especially well adapted, one constant is applicable to the stresses in all diagonals, another to the stresses in all verticals, and a third to the stresses in all chord members.

The derivation of the coefficients can be shown better by a numeric example than algebraically. It will be illustrated here in connection with a uniform load covering the entire bridge. This condition of loading produces maximum live-load stresses in the chord members. It also agrees with the usual assumption in regard to the distribution of the dead load. The live-load stress in any given web member is usually maximum when the load covers only a certain part of the bridge, as will be fully explained in Chapter VII, where the derivation of the coefficients for maximum live-load web stresses will also be shown.

Let it be required to find the coefficients for the eight-panel truss, Fig. 17, for a uniform live load of intensity w per truss, covering the entire bridge.

Diagonals. The load at every lower-chord joint is $W=wp$. As explained in the preceding article, the stress in each diagonal is found by multiplying the shear in the corresponding panel by h'/h. The shear in panel ab is $R=3.5W$. The shears in successive panels from bc to the center of the truss are $2.5W$, $1.5W$ and $0.5W$. The stresses in the diagonals are therefore the products of the coefficients shown in the figure by the constant Wh'/h. The character of each stress, found as in the preceding article, is shown by the sign + or −.

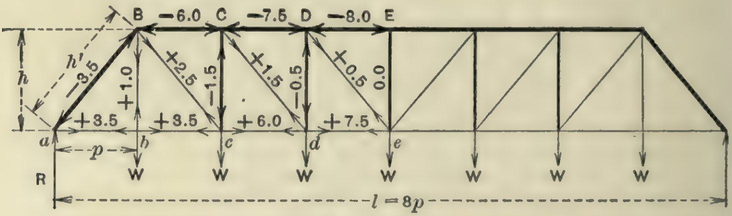


FIG. 17.

Verticals. Referring to joint b, stress $Bb=+W$. Referring to joints C and D, the stresses in the posts Cc and Dd are equal to the vertical components, $1.5W$ and $0.5W$, of the stresses in the diagonals Cd and De, respectively, which meet these posts in the *unloaded* chord (in this case the upper chord). From $\Sigma V=0$, applied to the forces at C, D, and E, stresses Cc and Dd act in the directions of the arrows, and are therefore compressive, and stress Ee is zero. The stresses in the verticals are therefore the products of the coefficients shown in the figure, by the constant, W.

Lower Chord. Considering joint a, from $\Sigma H=0$, stress ab is equal to the horizontal component of stress aB. This component may be found by multiplying the vertical component of stress aB, $3.5W$, i.e., the shear in panel ab, by p/h. Evidently stress bc = stress ab. Analyzing the stresses at joint c, from $\Sigma H=0$, stress cd is equal to stress bc plus the horizontal component of the stress in diagonal Bc. The latter component is found, as before, by multiplying the shear, $2.5W$, in panel bc by p/h. Thus stress $cd=6W\times p/h$. Similarly, stress $de=7.5W\times p/h$. For equilibrium, these stresses must act with respect to the successive joints in the directions of the arrows shown in the figure. The lower-chord stresses are therefore tensile, and their values are found by multiplying the coefficients in the figure by the constant Wp/h.

Upper Chord. Take a section (not shown) cutting BC, Cc, and cd. From $\Sigma H=0$, applied to the forces to either side of this section, stress BC = stress cd. For equilibrium these stresses must act in opposite directions, i.e., stress BC is compressive. Similarly, stress CD = stress de. These stresses might also have been found from $\Sigma H=0$, applied to the stresses meeting at the upper-chord joints. Thus from joint D, stress $DE=-8Wp/h$.

Summarizing, it is seen that the stresses are the products of the numeric coefficients shown in the figure and the following constants:

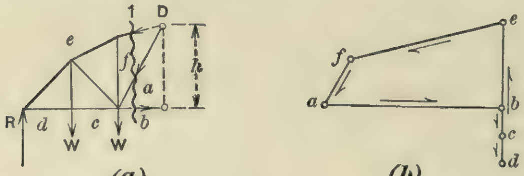
$$\left.\begin{array}{ll}\text{For the diagonals,} & W/h \times h' \\ \text{For the chords,} & W/h \times p \\ \text{For the verticals,} & W\end{array}\right\} \quad . \; . \; . \; . \quad (12)$$

By means of a slide rule the stresses may thus be quickly evaluated.

The values of the constants are seen to depend on the intensity of the load and the dimensions of the truss. The values of the coefficients depend, however, only on N, the number of panels in the truss. The latter may therefore be derived and conveniently tabulated for N ranging from 4 to 12, the higher value being rarely exceeded in trusses with horizontal chords. Such a table of coefficients for maximum live-load stresses in the web members will be given later (Art. 124). The coefficients for dead-load web stresses, and for chord stresses in general, may be so quickly found in any given case that tabulated values are of little or no advantage.

If W equals unity, the coefficients for the verticals represent the stresses in those members, and the coefficients for the diagonals represent the shears in corresponding panels. If h and p also equal unity, the coefficients for the chords represent the stresses in those members.

60. Graphic Methods. A partially graphic method applicable to the analysis of non-concurrent stresses was given in connection with Fig. 16, Art. 58. This article will be devoted to methods which are either purely graphic or essentially so.



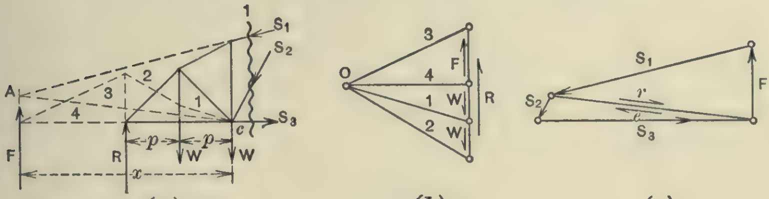
(a) (b)

FIG. 18.

First Method. The segment of a truss shown in Fig. 18 (*a*) is the same as that in Fig. 14 (*b*), Art. 57. Let it be required to find the stresses in the three members cut by section 1.

If M_D denotes the resultant moment of the external forces to the left of the section about D, i.e., the bending moment at D, stress $ab = M_D \div h$, in which M_D may be found graphically by the method for vertical forces explained in Art. 37 (*b*). After stress *ab* has been thus found, there remain only two unknowns, namely stresses *ef* and *fa*. These may therefore be found by drawing the force polygon *abcdefa*, Fig. (*b*). The character of the stresses may be ascertained in the usual way, from the arrows which point around the polygon in the same direction as the known forces.

Second Method. Referring to Fig. 19 (*a*) the position of the resultant, $F(=R-2W)$, of the external forces to the left of the section may be found graphically by drawing the equilibrium polygon 1, 2, 3, 4, whose segments are parallel to the corresponding rays in Fig. (*b*); or by computation, taking moments about c, $x=(R\times 2p-W\times p)/(R-2W)$. The force F and the three unknown stresses, S_1, S_2, and S_3, must be in equilibrium. The point of intersection of any two of these forces is therefore a point on the resultant, r, of the other two. Thus A is a point on the resultant of S_2 and S_3, and Ac is therefore the line of action of this resultant. The magnitude of r and S_1 may be found by drawing a force polygon, Fig. (*c*), of the forces meeting at A. The direction of the known force, F, indicates the directions of the arrows for S_1 and r. Stress S_1 is thus seen to be compressive. The equilibrant, e, of S_2 and S_3 acts in a direction opposite to r. Stresses S_2 and S_3 may now be found by drawing the force polygon for the forces meeting at c. The forces S_2 and S_3 must act in the same direction around the polygon as e. Stress S_2 is thus seen to be compressive and stress S_3, tensile.

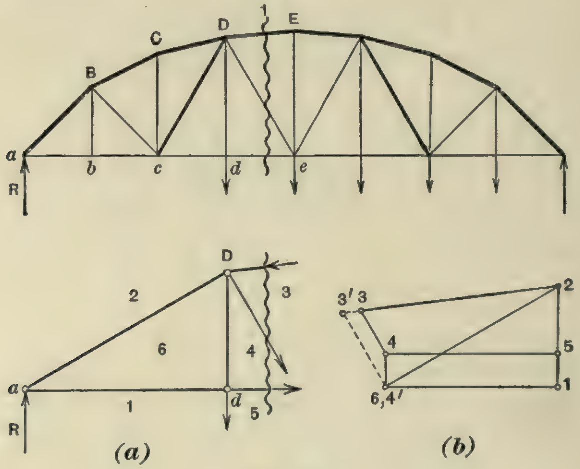


(*a*) (*b*) (*c*)

FIG. 19.

This method might conveniently have been applied to the problem in Fig. 16, Art. 58, after the resultant 5 of the external loads to the left of the section had been found.

Third Method. As will be seen later, the live-load stress in a given web member of a truss becomes maximum when the load extends over the greater segment of the truss to either side of the panel containing that member and partly into that panel. Thus, referring to Fig. 20, the live-load stress in diagonal *De* becomes maximum when the load extends from the right support to some point in panel *de*. The joints *b* and *c*, to the left of *d* will therefore be free from load. When the loading is placed in the correct position for maximum stress in *De*, the stress in no other member of the truss will be maximum, so that the values of the stresses in these other members are of no practical interest. It would therefore be a waste of time in this case to begin in the usual way at joint *a*, and to obtain the stresses in all the members to the left of *De*, merely for the purpose of finding stress *De*, which is the only stress desired. It is clear that if the segment of the truss to the left of section (1) is assumed to be replaced by the simple triangular frame shown in Fig. (*a*) the relative positions of the external forces will not be disturbed, and this imaginary frame will meet the requirements of stability. Hence the stresses in the three members cut by the section will be the same as before. The stress diagram, Fig. (*b*), may now be constructed by drawing the polygons for joints *a*, *d*, and *D* in the order stated, and 3–4 is then the desired stress (tension) in *De*.



(*a*) (*b*)

FIG. 20.

If no load exists at *d*, stress 4–6 is zero, and if the reaction *R* is then represented, as before, by 1–2, 3′–4′ will represent stress *De*.

61. Graphic vs. Algebraic Methods. The term 'algebraic,' applied to methods of stress analysis, is to be understood as referring to all non-graphic methods, although the processes involved may be numeric rather than algebraic.

The choice of the method best adapted to the determination of the stresses in any given case is governed partly by the conditions of the problem, and partly by individual preference.

In a fair comparison of the two general methods, it is to be assumed that each is used in the particular form best adapted to the problem under consideration. In connection with the algebraic method reasonable proficiency in the use of the slide rule is also to be presupposed.

Generally speaking, the graphic method is preferable in the analysis of roof trusses, first, because the inclinations of the members are apt to be rather irregular, and second, because the stresses from the principal classes of loading (dead, wind and snow) are based on loads assumed to act in the same fixed positions for every member of the truss.

On the other hand, in the analysis of bridge trusses the live load, as well as that part of the wind pressure which acts on the live load, must be treated as moving loads, and these loads must accordingly be assumed to occupy the particular positions in which they will produce the maximum stresses in each member in turn. Relatively, the graphic method is affected more unfavorably by these conditions than the algebraic method.

In the analysis of bridge trusses with horizontal chords, both for dead-load and live-load stresses, the algebraic method is, by general agreement, preferable to the graphic method, as being more expeditious, aside from its greater accuracy. In the case of bridge trusses with inclined chords, there is more room for difference of opinion. Some engineers prefer algebraic methods, others graphic methods. The choice may also be governed by the character of the live load, according to whether it consists of a series of concentrated weights, or whether it is uniformly distributed; and, if the latter, whether the stresses are to be found by the so-called 'exact' or by the 'conventional' method, which will be treated later. Practiced computers usually prefer algebraic methods also for bridge trusses with inclined chords. Sometimes methods which are partly algebraic and partly graphic may be used to advantage.

The judicious choice and rapid application of algebraic methods demands, generally speaking, a broader and surer command of principles than is required in the use of the ordinary graphic methods, besides practice in performing numeric operations accurately and quickly both with and without the aid of the slide rule. By the algebraic method the stresses may be determined to any desired degree of accuracy, whereas in the graphic method the accuracy of the results depends not only on the care used in the execution of the drawing, but on the scale that can be adopted without extending the drawing beyond practicable limits. Frequently the largest stress in a truss is several hundred times greater than the smallest. A scale suitable for the former would then be entirely too small for the latter, and vice versa. Thus if the ratio between these stresses is 400, then if the scale is such that the greatest stress is represented to a length of 10 ins. the smallest will appear to a length of only $\frac{1}{40}$ in., so that not only its magnitude, but even its character, will be uncertain, as will be illustrated later by an example (Art. 133). Graphic methods possess the advantage, however, of being, in a large degree, self-checking. Gross errors in stresses found by the algebraic method can often be detected by applying approximate check methods to be described later, or, after some experience, by what is little more than a mere inspection of the results to consider their reasonableness.

Algebraic methods are more general than graphic methods in the sense that they admit of the derivation of equations applicable to trusses of the same general type, under the same general conditions of loading. By the graphic method a new, independent diagram is required in each particular case, unless the trusses are similar geometric figures and similarly loaded, in which case the stress in any given member of one truss will bear the same fixed ratio to the stress in the corresponding member of the other. Algebraic and graphic methods will both be fully developed in later chapters, although the former, for reasons stated above, will be treated in greater detail.

CHAPTER VI

ROOF TRUSSES

62. The Elements of Roof Framing. The framing of an ordinary mill building is shown in skeleton outline in Fig. 1, in which the principal elements are also designated. The upper chords of the main trusses are called ' main rafters.' The height from the axis of the lower chord to the joint at the ' peak ' of the main truss is called the ' rise ' of the truss. The ratio of rise to span is called the ' pitch.' For roofs of this general character the most common value of the pitch is 1/4, which corresponds to a slope of 0.5 on 1. The term ' slope ' is preferable to ' pitch,' since it is free from ambiguity. Thus in the case of a shed roof, if a single trussed rafter is used, as is commonly done, the pitch, as above defined, would be the ratio of the rise to *twice* the span.

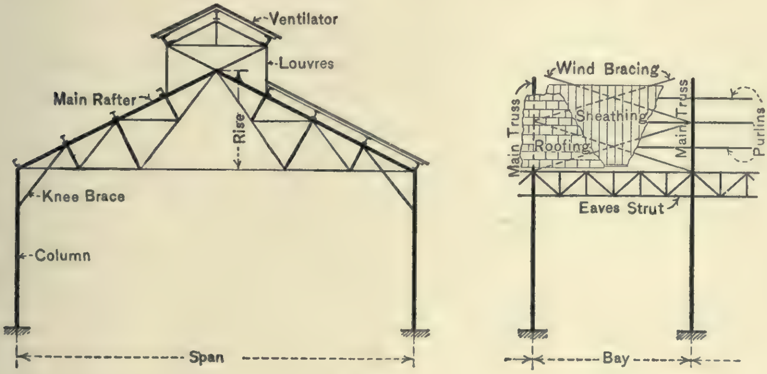


FIG. 1.

The main rafters support the ' purlins,' and the latter the ' covering.' The covering includes the wooden ' sheathing,' if any, and the ' roofing ' proper. If the roofing consists of corrugated steel, or any other material sufficiently stiff to be self-supporting between purlins, the sheathing may be omitted.

The distance between consecutive roof trusses is called a 'bay' or 'panel.' The former term is much used in practice, and avoids confusion with the term 'panel' applied to the subdivisions of the trusses themselves.*

The upper construction, to which the term 'ventilator' is applied in the figure, is also called a 'monitor.' The 'louvres' along the sides supply ventilation through a series of fixed horizontal slats, inclined transversely to exclude rain. Frequently sliding or pivoted sashes are used to serve for lighting as well as ventilating purposes.

Diagonal 'wind bracing' in the planes of the main rafters is commonly provided, as shown, in the end bays, and at intervals in the intermediate bays (usually in every fourth or fifth bay) of a series of trusses. The purlins are commonly utilized as struts in such a system of bracing. The provision of bracing in the plane of the lower chord depends largely on the prospective use of the building. Since the lower chord is in tension it has no tendency to bend sideways, and such lower bracing is usually omitted as an unnecessary restriction of head room except in panels in which it is required as lateral support for jib cranes or to prevent excessive vibration from shafting or machinery. In such cases longitudinal bracing may also be provided in the planes of the principal struts in the main trusses.

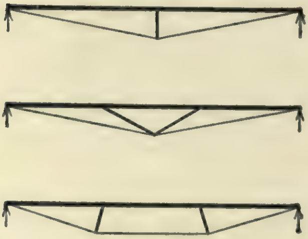


FIG. 2.

When the trusses are supported on columns, as in Fig. 1, rigidity in a longitudinal direction is secured by latticed 'eaves struts' riveted continuously between the columns directly below the eaves of the roof. Rigidity in a transverse direction is secured by the use of 'knee-braces' connecting the main trusses and the columns.

The spacing of the purlins depends on the character of the covering. It is desirable, in general, that the purlins should be placed only at the apexes of the truss. Frequently intermediate purlins are used, however. In that case the rafters are subject to bending, as well as axial stress, and must be proportioned accordingly by methods to be considered in Vol. III.

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* In English practice the term 'bay' is used exclusively in the sense in which 'panel' is used in American practice.

In the case of large trusses widely spaced the trusses are frequently connected by suitably spaced lattice girders supporting intermediate 'jack rafters,' which in turn support the purlins. The term 'jack rafter' is applied in general to single, independent rafters, unless they are given a more distinctive name by reason of their location. When the trusses are widely spaced it becomes economical to use 'trussed' purlins, of which some common types are shown in Fig. 2, instead of single I-beams, channels, Z-bars, etc.

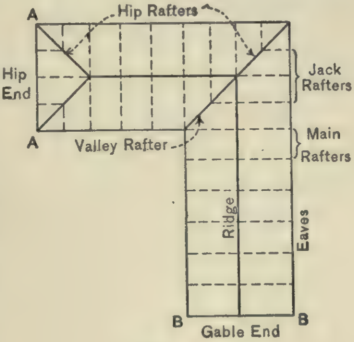


FIG. 3.

An L-shaped roof is shown in plan in Fig. 3. At *AA* the roof has a 'hip' end, and at *BB* a 'gable' end. The 'hip-rafters' and the 'valley-rafter' are shown by full lines; the main rafters and jack-rafters by dotted lines.

63. Standard Types of Trusses. Where æsthetic considerations or special requirements govern, roof trusses assume an almost endless variety of forms; but where they are designed to meet ordinary conditions of service the choice is commonly restricted to a few standard types.

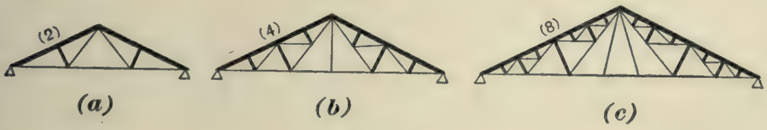
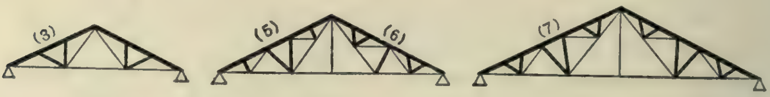


FIG. 4.

Fink Truss. For steel roofs up to spans of about 100 ft., the Fink truss, Fig. 4, is probably used more extensively than any other type, because of its combined economy and good appearance. In Figs. (*a*), (*b*) and (*c*) the rafters are divided into 2, 4 and 8 equal panels, respectively. The number of subdivisions is governed by the span and the required spacing of the purlins, the latter depending in turn on the character of the roof covering. By using pairs of struts, Fig. 5, instead of single struts, any desired intermediate subdivision may be effected. For longer spans, the middle member of the lower chord is usually supported at one or more points as in Figs. 4 (*b*) and (*c*) and 5 (*b*) and (*c*), not only to relieve the bending stresses in that member from its own weight, but also because of the frequent necessity of suspending machinery or other loads from the lower chord. To increase the overhead clearance the end members of the lower chord may be inclined as in Fig. 6. The Fink truss is also used in the modified form shown in Fig. 7.



(*a*) (*b*) (*c*)

FIG. 5.

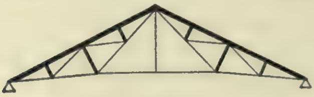


FIG. 6.

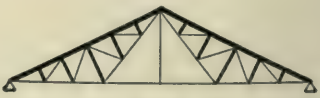


FIG. 7.

The economy of the Fink truss is due mainly to the fact that the struts in the web system are relatively short in comparison with the ties. Since the permissible working stresses for struts are considerably lower than for ties this condition is favorable to economy of material.

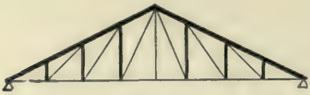


FIG. 8.

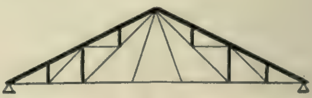


FIG. 9.

Pratt Truss. The Pratt truss, Fig. 8, in which the verticals are in compression and the diagonals in tension under vertical loading, is also frequently used. As the span and depth of truss increase it becomes economical to diminish the lengths of the verticals as in Fig. 9, thus practically converting the truss into a Fink truss with vertical struts.

Howe Truss. If the diagonals are inclined as in Fig. 10, i.e., in directions opposite to those in Fig. 8, the verticals are in tension and the diagonals in compression under vertical loading. This form of truss, known as a Howe truss, is especially well adapted to construction in wood, since it minimizes the amount of steel required for the tension members. The rafters and diagonals are made of wood, the verticals of steel, and the lower chord of either wood or steel.

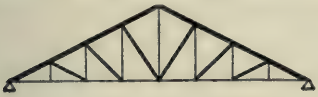


FIG. 10.

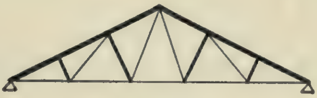
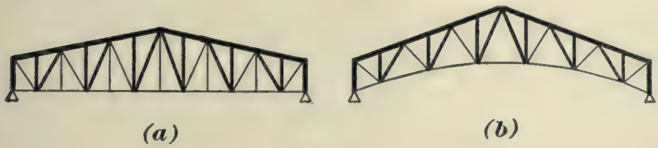


FIG. 11.

Triangular Truss. The triangular truss, Fig. 11, is much used for very short spans and also, in the form shown in Fig. 12, for large roofs with comparatively flat slopes. By changing the inclination of the lower chord at every apex, as in Fig. (*b*), good architectural effects may be produced. For steel roofs, the web system may also be made of the Pratt type. In this case, since the rafters do not intersect the lower chord, the diagonals must slope in directions opposite to those in Fig. 8, in order that they may be in tension under vertical loads (Art. 74). For wooden roofs, the web system may be made of the Howe type, with diagonals sloping in directions opposite to those in Fig. 10.



(a) *(b)*

FIG. 12.

Crescent Truss. For spans ranging from about 100 to 200 ft. the ' crescent ' or ' sickle ' truss, Fig. 13, with various types of web system, was formerly much used on account of its good appearance. It is less economical, however, than a truss of the general outlines shown in Fig. 12, and for long spans the three-hinged arch is now regarded as at once the best and most economical form of roof truss.

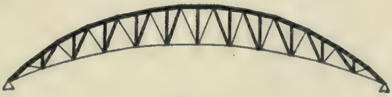


FIG. 13.

Cantilever System. Where the interior of a building is to be divided into a wide central space, unobstructed by columns but flanked by arcades, Fig. 14, the roof may be designed on the cantilever principle. The truss is a simple truss of span *l* supported at the free ends of the cantilever arms, *c*, which are connected to the anchor arms, *a*. The outside columns are subject to negative reactions from loads on the cantilever arms and central truss. If the resultant reactions are negative due provision must be made for their development by means of 'anchorages' of suitable weight and design. The Mines and Mining Building at the Chicago World's Fair furnished a noteworthy example of a roof of this type.

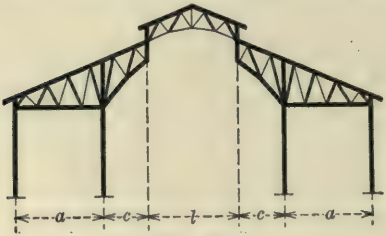


FIG. 14.

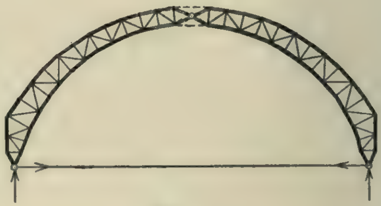


FIG. 15.

Three-hinged Arch. For roofs of very long spans, such as are used for armories, train sheds, and large assembly or exhibition halls, the three-hinged arch, Fig. 15, is the best type of structure, on grounds of both economy and appearance. In comparison with arches of certain other types it commends itself because its stresses are statically determinate.

The relative economy of three-hinged arches and cantilever roofs has not been thoroughly investigated. The arch is probably more economical for spans greater than about 150 ft. and possibly also for shorter spans. The choice is apt to be determined by the architectural requirements, apart from considerations of cost. The span of the Mines and Mining Building was 115 ft. between centers of inside columns.

A serious objection to the use of the arch, as ordinarily designed, is the diminution of the clear width of the room by the double width of the 'haunches' of the arch and the interference of the latter with the continuity of balconies along the side walls. This may be overcome by supporting the arch at the desired elevation on cantilever arms.*

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* Engineering News, Vol. 63, No. 18 (1910) and Vol. 65, No. 4 (1911).

64. Length of Bays. The general procedure in the design of a roof has been indicated in Art. 12. The economic length of bay depends not merely on the least weight of metal, but on the least combined cost of material, manufacture and erection. The economic length of bay increases with the span, but does not bear a fixed relation to the latter. With increasing length of bay the cost of the purlins increases and the cost of the trusses, collectively, decreases. For spans within about 100 ft., the bays of steel roofs are usually from 15 to 20 ft. long. The economic length depends partly on the span, and largely on the spacing of the purlins, and on whether the latter are trussed or consist of single pieces. For long-span roofs the economic length of the bays is greatly influenced by the general character of the design. This subject will be treated in more detail in Vol. III, relating to the design of framed structures.

65. The Dead Load consists of the combined weight of (1) the roof covering; (2) the purlins; (3) the main trusses (including auxiliary rafters, if any) and bracing; and (4) special loading, such as machinery, suspended ceilings, floors, etc.

Roof Covering. The character and weight of the roof covering should be first determined, since it directly affects the spacing and design of the purlins. The weight of the roof covering, per square foot of horizontal projection of roof, usually varies from less than 2 lbs. (for No. 22 corrugated steel sheets without sheathing) to 15 lbs., although in exceptional cases it may be as great as 30 lbs. or more. Its weight in any given case is readily determined, according to the materials and thicknesses to be used, from data easily accessible in standard engineering and manufacturers' handbooks.

Purlins. After the weight of the roof covering has been found, the purlins may be designed for that weight, their now (assumed) weight, and the assumed wind pressure and snow load, if any. The weight of steel purlins, per square foot of horizontal projection of roof, usually ranges from 2 to 3.5 lbs. for spans within 100 ft. The weight of the purlins may be reduced by trussing (Fig. 2), but their cost per pound is thereby increased. For long-span roofs this weight may reach 6 lbs. or more. After the purlins have been designed their weight should be determined, and if it differs materially from the weight assumed in the calculation of stresses, the latter should be revised and the design modified if necessary.

Main Trusses. Many formulas for finding the weights of roof trusses have been proposed. The variables are so numerous, however, and the influence of certain factors is so great, that no formula that is at once simple, accurate and generally applicable, can be devised, as will be shown later (Art. 76). The weight per square foot of horizontal projection of ordinary steel roof trusses, supported on walls, and having the usual slope of 0.5 on 1 (1/4 pitch) generally ranges from about 2 to 2.5 lbs. for a span of 30 ft. to about 5 to 6 lbs. for a span of 100 ft.; and within these span limits the weight of the bracing usually runs from 0.3 to 0.8 lb. For roofs of steeper slopes, the weight of the trusses is apt to decrease somewhat, and for flatter roofs the weight becomes somewhat greater. The total weight of trusses, purlins and bracing for roofs of the same span may easily vary 25 per cent or more, from circumstances of which no single simple formula can take proper account.

The total weight of metal, per square foot of area covered, of some of the largest roofs of modern design is as follows:

Building.	Span, c. to c.	Purlins and Rafters, lbs. per sq. ft.	Total Weight of Steel, lbs. per sq. ft.
Mfrs. and Lib. Arts Bldg., World's Columbian Exposition..........	368 ft.	5.3 (purlins only)	21.0
Penna. R. R. Trainshed, Philadelphia, Pa.....................	300 ft. 8 ins.	7.6	39.1
Phila. and Read. R. R. Trainshed, Philadelphia, Pa..............	259 ft. 8 ins.	6.8	34.1
Penna. R. R. Trainshed, Jersey City, N. J....................	252 ft. 8 ins.	3.9	27.9

The weight of the trusses and bracing may be assumed approximately in the light of the above data, and combined with the known weight of the roof covering and purlins, thus greatly reducing the percentage of probable error in the assumed total dead load. Since the truss members are finally proportioned for combined stresses from dead load, wind load and snow load, if any, the percentage of error in the stresses due to inaccuracies in the assumed weight of trusses and bracing is further reduced to a small and usually negligible minimum. After the trusses and bracing have been designed, their weight is computed, and if it differs materially from the assumed weight, the stresses may be revised and the dimensions of the members altered accordingly.

Special Loading. If the trusses are subject to special loading of the kind referred to at the beginning of this article the probable effect of this loading on the weight of the trusses must be judged according to the circumstances of the case. The assumed weight of the trusses must be correspondingly increased, subject to revision in the light of the computed weight on completion of the design. In judging the effect on the weight of the trusses of a given ratio of increase of total loading, it is to be observed that the weight increases at a somewhat smaller ratio than the loading.

66. Wind Pressure. Theoretically the pressure p, in pounds per square foot, on a plane surface normal to the direction of flow of a fluid having a relative velocity v, in feet per second, is equal to the weight of a vertical column of the fluid, having a cross-section of one square foot, and a height h, in feet, equal to that through which a freely moving body must fall to acquire the velocity v. If w denotes the weight of the fluid in pounds per cubic foot,

$$p = wh = \frac{wv^2}{2g}. \qquad \text{(1)}$$

For air at a temperature of 32° F. and at a barometric pressure of 760 mm., $w = 0.081$. Letting $g = 32.2$,

$$p = 0.00126\, v^2. \qquad \text{(2)}$$

If V denotes the velocity of the wind in miles per hour, $v = 1.47\ V$,* whence Eq. (2) becomes

$$p = 0.0027\, V^2. \qquad \text{(3)}$$

Expressing this equation in the general form $p = cV^2$, in which c is an empiric coefficient, experimental investigations of wind pressures on thin plates indicate that the actual value of this coefficient is greater than 0.0027. Its value is found to be somewhat affected by the velocity of the wind, and by the size and shape of the plate, but chiefly by the formation of a partial vacuum at the back of the plate which tends to increase greatly the resultant pressure in the direction of the wind.*

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* It is interesting to note that the velocity in feet per second is approximately 1.5 times the velocity in miles per hour.

Experiments on wind pressures have usually been conducted by means of whirling machines with peripheral vanes revolving in still air, the conclusions, especially those of earlier dates, being liable to error from many sources. From Rouse's experiments on windmills, reported by Smeaton in 1759, $c=0.00492$, or approximately, $c=0.005$, a value largely quoted to this day. The value, $c=0.00492$, was confirmed by Duchemin's experiments made in about 1829, " for ordinary air pressures on thin stationary plates."

From experiments made by Langley in 1888 and 1890 on a large scientifically designed whirling machine, $c=0.0039$, this coefficient having been deduced from observations on plates of from 0.25 to 1 sq.ft. area, moving at velocities of from 10 to 70 miles per hour in circles 60 ft. in diameter. From observations of actual wind pressures, by Marvin in 1890, on surfaces 4 and 9 sq.ft. in area, $c=0.004$. These and other experiments of modern date indicate that the pressure on *thin plates of moderate size* from wind blowing at a *uniform velocity*, may be found approximately by the formula

$$p=0.004\,V^2, \qquad (4)$$

in which p denotes the pressure in pounds per square foot and V the velocity in miles per hour. The values of V by this formula for certain even values of p, and vice versa, are given in the following table:

p lbs. per sq. ft.	V miles per hr.	V miles per hr.	p lbs. per sq. ft.
10	50	40	6.4
15	61	50	10.0
20	71	60	14.4
25	79	70	19.6
30	87	80	25.6
35	94	90	32.4
40	100	100	40.0

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* A valuable report on *Wind Pressures*, by Captain (now General) W. H. Bixby, U. S. Corps of Engineers, published in 1894 as Appendix C to the Report of Board of Engineer Officers as to *Maximum Span Practicable for Suspension Bridges*, contains an excellent summary of information on this subject. The data in this article have been drawn partly from that source.

From an engineering standpoint, however, it is of the greatest importance to note that the wind does not blow at a uniform velocity, and that the simultaneous intensities of pressure on various parts of a large surface exhibit great differences. It follows, therefore, that *the maximum average intensity of pressure decreases as the area of the surface increases.* Thus from observations made from 1884 to 1890 at the site of the Forth bridge, on a surface having an area of 300 sq.ft., it was found that the maximum average intensities of pressure during four heavy storms which blew squarely against this surface, were from 23 to 60 per cent less than the maximum average on surfaces having an area of only 1.5 sq.ft. The maximum average pressure on the large surface was 27 lbs. per square foot, and that on the small surface 41 lbs. per square foot.*

The effect of pressures on comparatively small surfaces may be greatly intensified by reason of the fact that the velocity of the wind is subject to continual rapid fluctuations. Marvin found that the magnitude of these variations amounted to as much as 35 per cent of the mean pressure, and that they occurred within a second or two. Langley observed fluctuations varying 20 to 50 per cent from the average pressure, and found that they became greater at higher velocities. If the rate at which these fluctuations occur happens to be synchronous, or approximately so, with the time of vibration of a structure or any of its component parts, the effect becomes cumulative. This circumstance has been advanced in explanation of certain remarkable effects of wind storms.

Although meteorological observatory records,† as well as computations based on the effects of wind storms in extreme cases, such as the overturning of a light locomotive in East St. Louis in 1871,‡ indicate wind pressures as high as 80 to 93 lbs. per square foot, the path of greatest violence of such storms is relatively narrow and not many miles in length. Thus C. Shaler Smith, from an examination of the tracks of several tornadoes, found but a single instance in which the pathway wherein the computed pressure exceeded 30 lbs. per square foot was more than 60 ft. wide.* By Eq. (4) a pressure of 30 lbs. per square foot corresponds to a velocity of about 87 miles per hour.

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* *The Forth Bridge,* Engineering, Feb. 28, 1890.

† As, for example, the records at Bidston Observatory, near Liverpool, quoted in Captain Bixby's report.

‡ *Wind Pressures upon Bridges,* by C. Shaler Smith, Trans. Am. Soc. C. E., Vol. X, 1880.

Practical Conclusions. It is economically if not physically impracticable—and it would be ill-advised, if practicable—to design roofs and buildings in general to withstand the attacks of such rare visitations as great tornadoes. In good practice the usual allowance for pressures on surfaces normal to the direction of the wind is 30 lbs. per square foot, which is about the strength of ordinary windows. This allowance may be regarded as amply sufficient for small roofs and buildings except in very exposed situations, for which 40 lbs. may be assumed. In built-up districts and for buildings which present very large surfaces to the wind the assumed pressure may be safely reduced to 25 lbs. or less. Thus for high buildings the building laws of Philadelphia require an allowance of 25 lbs. for the tenth story, an increase of 2.5 lbs. for each successive higher story, and a like reduction for each successive lower story.

67. Wind Pressure on Inclined Surfaces. Since the friction of air on comparatively smooth surfaces is very slight it may be assumed without sensible error that wind pressure can be exerted only in a direction normal to a surface. In estimating wind pressures, the direction of the wind is always assumed to be horizontal. The corresponding normal pressure on surfaces inclined to the vertical can be found only by experiment. By setting the vane in a whirling machine at any desired angle α, Fig. 16, with the tangent to its circular path of motion, the tangential component, p_h, of the pressure p_n, normal to the vane, is found experimentally, whence

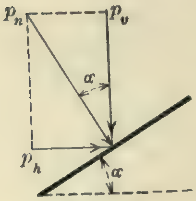

FIG. 16.

$$p_n = \frac{p_h}{\sin \alpha} = p_h(\sin \alpha)^{-1} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From experiments made by Hutton in 1787–88, on thin 4 by 8-in. plates, it was found that if p denotes the pressure on a plane normal to the direction of motion ($\alpha = 90°$), the component, p_h, of the pressure on the same plane set at any smaller angle α has the value

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* *Wind Pressures upon Bridges*, by C. Shaler Smith, Trans. Am. Soc. C. E., Vol. X, 1880.

$$p_h = p(\sin\alpha)^{1.842\cos\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, from Eq. (5),

$$p_n = p(\sin\alpha)^{1.842\cos\alpha - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From his experiments on small thin plates in about 1829, Duchemin deduced the formula,

$$p_n = p\frac{2\sin\alpha}{1+\sin^2\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The results of Langley's experiments (1888–90), previously referred to, were found to agree within 2 per cent with the values deduced from Duchemin's formula, and also served to confirm the validity of the assumption that the resultant pressure on an inclined plate is exerted in a direction normal to the surface. The values of the ratio p_n/p as well as the values of p_n for $p = 30$ lbs. per square foot for different values of α, are given in the following table, both for Hutton's and Duchemin's formulas:

TABLE I

WIND PRESSURES NORMAL TO INCLINED SURFACES

Angle of Inclination to Horizontal Plane, Degrees.	p_n/p		p_n for $p=30$. lbs. per sq.ft. of surface.	
	Hutton.	Duchemin.	Hutton.	Duchemin.
5	0.13	0.17	3.9	5.1
10	0.24	0.34	7.2	10.2
20	0.46	0.61	13.8	18.3
21.8 (slope 0.4 on 1)	0.50	0.65	15.0	19.5
26.6 (slope 0.5 on 1)	0.59	0.75	17.7	22.5
30	0.66	0.80	19.8	24.0
33.7 (slope ⅔ on 1)	0.73	0.85	21.9	25.5
40	0.83	0.91	24.9	27.3
45 (slope 1 on 1)	0.90	0.94	27.0	28.2
50	0.95	0.96	28.5	28.8
60	1.00	0.99	30.0	29.7
90	1.00	1.00	30.0	30.0

For angles below 50° Duchemin's formula is seen to give higher values than Hutton's. For the common slope of 0.5 on 1, the difference is 27 per cent of the smaller value; for angles above 50° both formulas give practically the same result.

It is important to note that the formulas of Hutton and Duchemin are based on observations on *thin plates.* The pressure on the flat face of a body may be reduced about 20 to 30 per cent according to the size and shape of the surface to the rear. The pressure on a body is affected much more, however, by the shape of the surface exposed to the wind. Thus the pressure on a cylindrical or conical surface, such as that presented by a chimney or a lighthouse, varies, according to different experiments, from about 50 to 60 per cent of that on a thin plate whose size equals the diametral section of these solids, and the pressure on concave surfaces may exceed that on plane surfaces equal to their projected areas by 75 per cent or more.

68. Wind Pressures on Roofs and Buildings. Hutton's formula was formerly much used in estimating the pressures normal to the inclined surfaces on the windward side of a roof, but Duchemin's formula, confirmed by Langley, is now regarded as more reliable. As a matter of fact, however, both formulas are based on experiments on small thin plates for which the conditions are wholly dissimilar from those applicable to roofs and buildings. Experiments by Kernot in 1893, on small models of roofs,* tend to show that the above method of estimating wind pressures is inapplicable to roofs covering buildings enclosed in walls. In such cases the upward deflection of the air on the windward side tends to reduce greatly the pressure on this side of the roof, while the formation of a partial vacuum on the leeward side induces a negative pressure, or uplift, on the roof on that side.

Experiments by Irminger in 1894 † were so arranged that the positive and negative pressures on different parts of the model could be quantitatively determined. Observations on thin plates showed that the resultant pressure was due in a larger measure to the suction on the leeward side than to the pressure on the windward side. The effect of suction in percentage of total effect varied from 100 for α up to 5°, to 45 for $\alpha = 90°$. Experiments on a small model of a roof supported on walls showed a distribution of pressure as indicated by the shaded areas in Fig. 17, p denoting the resultant pressure on a thin plate of height h. The reduction of pressure on the windward side of the roof due to the upward deflection of the air from the windward wall is clearly apparent. The effect of the suction on the leeward side of the roof is seen to be over three times as great as that of the pressure on the windward side. The resultant, $R(=0.38p)$, of the wind forces on the roof proper was found to be 3½ times as great as the total pressure ($0.11p$) on the windward side, and to have an upward inclination tending to lift the roof from its supports, as has been frequently observed in the destructive effects of wind storms. The resultant (not shown) of the forces $0.11p$ and $0.42p$ on the windward side of the model has a downward inclination, while the resultant of the two equal forces $0.36p$ on the leeward side is greater and inclined upward. The resultant outward pressure on the leeward side of a building may be greatly intensified by the blowing in of the windows on the windward side. Thus during the St. Louis tornado in 1896, "in numerous instances the windows were blown in on the windward side, while the entire wall was blown out on the leeward side." *

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* *Engineering Record,* Vol. XXIX, No. 11, 1894.

† *Engineering News,* Vol. XXXIII, No. 7, 1895.

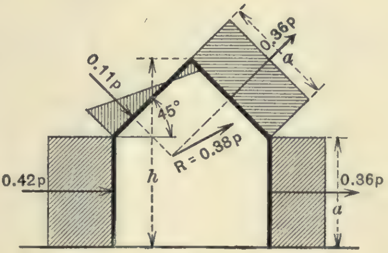

FIG. 17.

Experiments on small roof models by Stanton in 1903 † gave results substantially in accord with those of Irminger for a rafter angle of 45° with the horizontal. For an angle of 30°, the pressure on the windward side of the roof became negative in the vicinity of the lower as well as the upper edge, and remained positive only over about the middle third of the surface, the resultant pressure on that side of the roof becoming practically zero.

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* *Wind Pressures in the St. Louis Tornado*, by Julius Baier, Trans. Am. Soc. C. E., Vol. XXXVII, 1897.

† *On the Resistance of Plane Surfaces in a Uniform Current of Air*, by T. E. Stanton, Proceedings of the Institution of Civil Engineers, Vol. CLVI, 1904.

Practical Conclusions. It is seen from the foregoing that existing knowledge as to the intensity of wind pressures, and as to its distribution over large surfaces and on enclosed structures, is very inadequate, and that it is based mostly on experiments with small models. The experiments of Kernot, Irminger and Stanton were made on much too small a scale to admit of quantitative deductions applicable to conditions in practice. They are valuably suggestive, however, in calling attention to conditions which were previously not generally or adequately recognized. For the present it would hardly seem prudent to assume the normal pressures on the windward side of a roof resting on walls at lower values than those deduced from Duchemin's formula for $p=30$ lbs. per square foot, as given in Table I of the preceding article. This pressure is much greater than that found in Irminger's and Stanton's experiments on small models of roofs resting on walls. If therefore the trusses in such cases are designed to resist also a negative pressure, or uplift, of say half the downward pressure by Duchemin's formula, the requirements will probably be adequately met. It is obviously important that the trusses should be properly anchored to the walls, and that the walls should be further secured laterally by connections with the floor girders.

In view of the uncertainty as to the intensity and distribution of wind pressures, refinements in the computation of wind-load stresses are unwarranted. It is very important, however, that the individual truss members should be so designed as to be capable of developing compressive as well as tensile stresses in order to avoid the danger of collapse through the buckling under compression of a flexible member designed to act only in tension.

69. Snow Loads. The weight of freshly fallen snow varies from about 5 to 12 lbs. per cubic foot, according to its dampness. Saturated snow may weigh 45 lbs. or more per cubic foot, but under conditions producing such saturation snow on sloping roofs would be largely washed away. At about the latitude of New York and at moderate elevations, 20 lbs. per square foot of horizontal projection is a reasonable allowance for snow on roofs having a slope of 0.5 on 1. This may be increased to 30 lbs. for roofs which are nearly flat, and decreased to 10 lbs. for slopes of 1 to 1.5 on 1, according to the smoothness of the roof—to make allowance for sleet rather than snow, since the latter will attain no appreciable depth on roofs of such slopes. For more northern or mountainous districts these allowances may be increased about 50 per cent, and for more southern districts near the fringe of the snow belt they may be decreased to a minimum of 25 per cent of the above values.

70. Combined Wind and Snow Loads. The assumption on which the combined effect of wind and snow loads should be estimated is largely a matter of individual judgment, which, in the absence of reliable data, can be little better than guess-work. The wind pressure is usually assumed at 30 lbs. per square foot of vertical surface, and the snow load for roofs having a slope of 0.5 on 1, at 15 to 20 lbs. per square foot of horizontal projection. The combined effect of these loads is usually estimated on one of the following assumptions:

(*a*) A uniform vertical load of 30 lbs. per square foot of horizontal projection is assumed over the whole roof. Roofs of spans up to about 100 ft. are commonly designed on this basis, although it is irrational, especially, as will be seen later, for roofs supported on columns stayed by knee-braces or other means.

(*b*) The stresses are determined separately for maximum snow load over the entire roof, and maximum wind pressure in either direction, reduced to an equivalent pressure, normal to the roof surface, by Duchemin's formula, and the maximum combined stresses are then found by summation. The implied assumption that the snow on a roof will remain undisturbed in a wind storm blowing at a velocity of nearly 90 miles an hour is clearly untenable.

(*c*) The wind pressure is assumed as in (*b*), and the maximum snow load is assumed on the leeward side only. This seems unreasonable, at least for roofs on buildings with enclosed sides, in view of the suction created on the leeward side of such roofs by great wind storms.

(*d*) The wind pressure is assumed as in (*b*), and one-half the maximum snow load is assumed over the entire roof. While such a condition could hardly apply to snow, this assumption makes suitable provision for sleet.

Another reasonable assumption would be the combination of maximum snow load over the entire roof and a wind pressure of the maximum intensity consistent with that condition. If the snow is regarded as partially protected by a coating of ice the following assumption is probably warranted:

(*e*) The wind pressure assumed as in (*b*), but at one-half its maximum intensity, and maximum snow load over the whole roof.

As will be seen from the example in the following article the stresses based on assumptions (*d*) and (*e*) do not differ greatly for roofs of the usual slope of 0.5 on 1, and for the commonly assumed values for wind pressure and snow. For very large roofs of varying slopes it seems desirable that the stresses should be combined on both assumptions with a view of proportioning each member for its maximum stress. For ordinary roofs either assumption may be made, although in northern districts the approximate realization of assumption (*e*) is much more probable than that of assumption (*d*). It is also to be observed that the stresses by assumption (*d*) are sometimes lower than those from the dead load and the maximum snow load combined (see Art. 71, Table II, member 4–5), in which case the members in question should evidently be proportioned for the latter.

Roofs have not infrequently been designed on the incorrect assumption that the wind pressure acts in a horizontal direction on inclined surfaces.

In practice account is seldom taken of the possible effect of uplift from wind, except to the extent of avoiding the use of flexible members and in the anchorage of the trusses at their supports.

71. Stresses in a Pratt Truss. The stresses in the Pratt truss, Fig. 18, will be determined on the basis of the following data:

Span, 50 ft.; slope of rafters, 0.5 on 1; length of bays, 15 ft.

Dead Load, 18 lbs. per square foot of horizontal projection of roof, made up as follows: covering, 11 lbs.; purlins, 3 lbs.; trusses and bracing, 4 lbs.

Snow Load, 20 lbs. per square foot of horizontal projection of roof.

Wind Load, 30 lbs. per square foot of vertical projection of roof.

For convenience all panel loads and stresses will be expressed in thousands of pounds.

Dead-load Stresses. In practice, the dead load of roof trusses is always assumed as uniformly distributed, and as applied wholly at the upper-chord joints. Strictly speaking, the weight of the trusses should be considered as applied partly at the lower-chord joints, but the assumption named introduces only very slight errors on the side of safety in certain web members —in the present case, the verticals—and is sufficiently accurate for practical purposes.

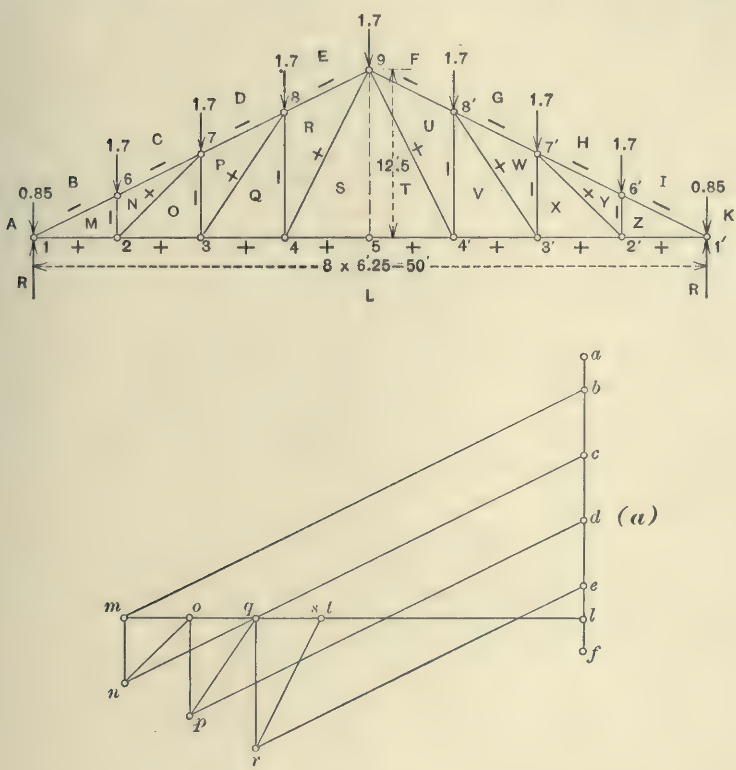


FIG. 18.

The panel length is 6.25 ft. and the dead apex loads are 6.25×15×18=1688 lbs. In terms of thousands of pounds, the apex loads may be assumed with sufficient accuracy at 1.7. The reaction at each support is 1.7×4=6.8. The half-panel loads, 0.85, at the end joints 1 and 1′ evidently do not affect the stresses in the truss.

The dead-load stress diagram, Fig. (*a*), is constructed, as explained in Art. 52, by first laying off the external forces, and then drawing the force polygons for successive joints, beginning at joint 1. This figure was originally drawn to a scale of 2000 lbs. to the inch, and the truss diagram to a scale of 5 ft. to the inch. Since the stresses in symmetrically disposed members are equal, the stress diagram is developed only for the left half of the truss. The values of the stresses appear in Table II of this article. As a check on the accuracy of the construction, the stress in the middle lower-chord member 4-4′ will be computed. As shown in Art. 55 (Second Method), this stress is equal to the end reaction, in the case of a truss whose rafters have a slope of 0.5 on 1. Thus stress 4-4′ = 6.8, which agrees with the value found graphically.

Snow-load Stresses. The snow load is assumed at 20 lbs. per square foot, and the dead load at 18 lbs. per square foot. Hence the snow-load stresses may be found by multiplying the dead-load stresses by 1.11. These stresses appear in Table II.

Wind-load Stresses. From Table I, Art. 67, the wind pressure normal to the surface of a roof, for a slope of 0.5 on 1 and $p = 30$ lbs., is 22.5 lbs. per square foot of inclined surface. The rafter panels are 7 ft. long, and the wind apex loads are $7 \times 15 \times 22.5 = 2.4$ thousand pounds.

To illustrate the influence on the wind-load stresses of the conditions at the supports it will be assumed, first, that both ends of the truss are fixed, and second, that one end is fixed and the other end is supported on rollers.

(*a*) *Both Ends Fixed.* The horizontal components of the reactions are in this case statically indeterminate (Art. 23) and they will be assumed to be equal. Referring to Fig. 19, the resultant wind pressure is $2.4 \times 4 = 9.6$. The horizontal component of this resultant is $9.6 \times 12.5/28 = 4.29$. The horizontal components of the reactions are therefore 2.15. The vertical components of the reactions may be best found by computation; i.e., by combining the reactions from the vertical and horizontal components of the resultant wind pressure, considered separately. The vertical component of this resultant is $9.6 \times 25/28 = 8.57$, which produces an upward reaction of $8.57 \times 3/4 = 6.43$ at the left support. The horizontal component, 4.29, produces a negative reaction of $4.29 \times 6.25/50 = 0.54$ at the left support. The vertical component of the reaction at the left support is therefore 6.43 − 0.54 = 5.89, and that at the right support, 8.57 − 5.89 = 2.68. The resultant reactions, *R* and *R′*, may be found by combining the components graphically.

The stresses in the members represented by broken lines are zero. In Fig. (*a*), *abcdefga* is the force polygon of the external forces. The values of the reactions may be checked by scaling *fg* and *ga*, drawn parallel to *R′* and *R*, respectively.

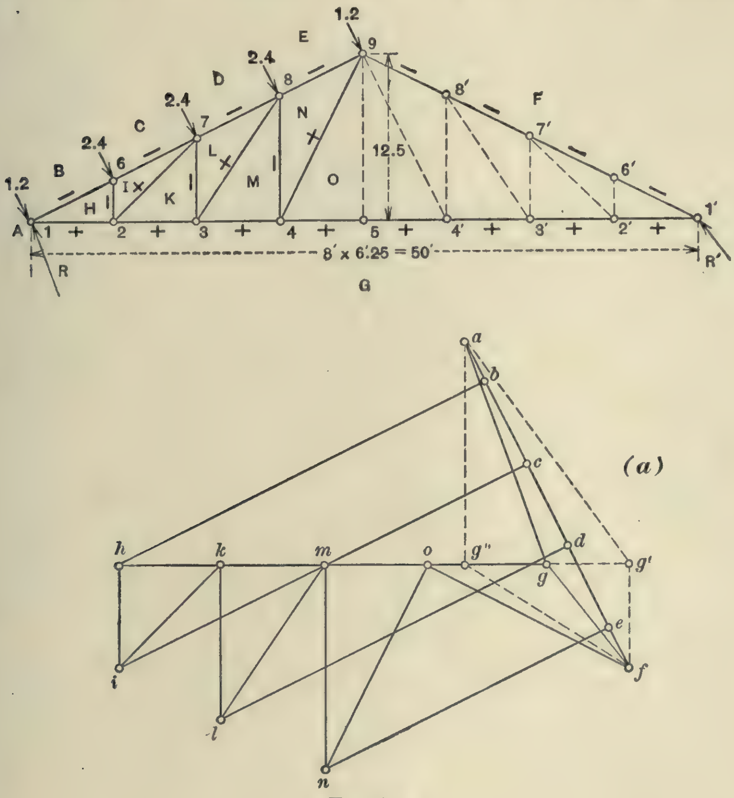
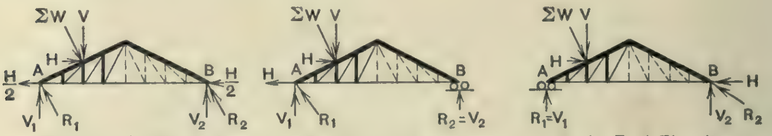


FIG. 19.

The stress diagram is developed in the usual way, beginning at joint 1 and ending at joint 1′. The closing line, drawn from *f* parallel to the right-hand rafter, must pass through the point *o*, which serves to check the accuracy of the drawing. It is seen that the stress *fo*, in the leeward rafter, is less than the stresses *bh, ci, dl* and *en* in the windward rafters, and that the stress *go*, in the lower chord on the leeward side, is the smallest stress in that chord. Hence the stresses in the windward half of the truss govern in all cases. The values of these stresses are given in Table II.

(*b*) *One End Fixed, Other End on Rollers.* Let it be assumed that the left end of the truss, Fig. 19, is fixed and the right end on rollers. The reaction at the roller end is then vertical. The stresses must be analyzed for the wind blowing first, from the left, and second, from the right. For the second case the wind may be assumed to act as shown in the figure with the above conditions at the supports interchanged.

In Fig. 20, let ΣW represent the resultant of the wind pressure, and H and V its horizontal and vertical components. Let V_1 and V_2 denote the vertical components of the reactions at the left and right supports respectively. Since the moments of the horizontal components, H and $\frac{1}{2}H$, of the reactions are zero about both supports, it is seen, by taking moments about the several supports, that the values of V_1 and V_2 are unaffected by the conditions at the supports.



Both Ends Fixed
(*a*)

Left End Fixed
(*b*)

Right End Fixed
(*c*)

FIG. 20.

The stresses for the conditions in Fig. 20 (*a*) are represented in the stress diagram, Fig. 19 (*a*). The effect on these stresses of the changes in the reactions shown in Figs. 20 (*b*) and (*c*) will now be considered.

Graphically. It follows from the foregoing that the reactions in Fig. 20 (*b*) may be represented graphically in Fig. 19 (*a*) by *fg′* and *g′a*, and the reactions in Fig. (*c*) by *fg″* and *g″a*. The point *g* is merely shifted to the right and left by the distances *gg′* and *gg″* $(=\frac{1}{2}H)$, respectively. The former movement serves to increase, and the latter to decrease the lower-chord stresses by $\frac{1}{2}H = 2.15$ or, say, 2.2, the stresses in all other members remaining unaffected. If the slope of the reaction *fg″* is flatter than that of *fo* drawn parallel to the rafter, g'' will lie to the left of *o*, and stress og'' in the lower chord on the leeward side is then compressive.

By Moments. Taking successive sections (not shown) cutting three members, and considering the forces to the right of such sections, Fig. 20 (*b*) differs from Fig. (*a*) only by the omission of the horizontal force $\frac{1}{2}H$ at the right support. If the stress in each member is found by taking moments of the forces to the right of the section about the intersection of the other two members as a moment center, it is seen that the moment centers of all the web members in the left half of the truss lie at A, and that the moment centers of all the upper-chord members lie in the lower chord. Since the moment of the force $\frac{1}{2}H$ is zero about these moment centers, it follows that the stresses in the web members and rafters are unaffected by the conditions at the supports. The moment centers of the lower-chord members lie at the rafter-joints. But the lever arm of $\frac{1}{2}H$ about the moment center of any lower-chord member is the same as the lever arm of that member. Hence the force $\frac{1}{2}H$ at the right support in Fig. (*a*) produces a compressive stress of a like amount in every lower-chord member. The absence of this force in Fig. (*b*) will therefore increase the tension in these members for the conditions shown in Fig. (*a*), by $\frac{1}{2}H$. Similarly the effect of increasing the horizontal force at B to H, as in Fig. (*c*), is a decrease of $\frac{1}{2}H$ in the lower-chord stresses in Fig. (*a*).

The maximum wind-load stresses in the truss with one end fixed and the other on rollers are given in Table II.

Combined Stresses. In Table II the wind and snow-load stresses are combined according to assumptions (*d*) and (*e*) in the preceding article and added to the dead-load stresses. The maximum stresses are distinguished by asterisks. The stresses resulting from these assumptions on the basis of 30 and 20 lbs. per square foot for wind and snow loads, respectively, and for a slope of 0.5 on 1 do not differ greatly as a rule. Assumption (*d*) is seen to govern for the web stresses and (*e*) for the chord stresses excepting member 1–2. The percentage of difference is about 9 in several cases and in one case (member 4–5, both ends fixed) as much as 16. The average percentage of difference is about 6.5. In a single instance, that of member 4–5, the stress (+13.9) by assumption (*d*) for the truss fixed at both ends, is

TABLE II

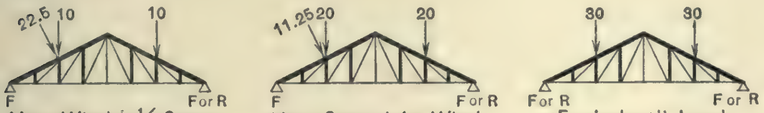
STRESSES IN A PRATT ROOF TRUSS

Position.	Member.	Dead Load.	Snow Load.	Wind Load.		Combined Stresses.				Load of 48 lbs. per sq.ft. over whole Roof.	Wind Uplift.	
						Dead + Wind + ½Snow		Dead + Snow + ½Wind				
				Both Ends Fixed.	One End Fixed.	Both Ends Fixed.	One End Fixed.	Both Ends Fixed.	One End Fixed.		Both Ends Fixed.	One End Fixed.
Lower chord	1–2	+11.8	+13.1	+11.3	+13.5	+29.7	+31.9*	+30.6*	+31.7	+31.5	−5.7	−6.8
	2–3	+10.1	+11.2	+ 8.7	+10.9	+24.4	+26.6	+25.7*	+26.8*	+26.9	−4.4	−5.5
	3–4	+ 8.5	+ 9.4	+ 6.0	+ 8.2	+19.2	+21.4	+20.9*	+22.0*	+22.7	−3.0	−4.1
	4–5	+ 6.8	+ 7.6	+ 3.3	+ 5.5	+13.9	+16.1	+16.1*	+17.2*	+18.1	−1.7	−2.8
Upper chord	1–6	−13.2	−14.7	−10.8		−31.4		−33.3*		−35.2	+5.4	
	6–7	−13.2	−14.7	−12.0		−32.6		−33.9*		−35.2	+6.0	
	7–8	−11.3	−12.6	−10.2		−27.8		−29.0*		−30.1	+5.1	
	8–9	− 9.4	−11.6	− 8.4		−23.6		−25.2*		−25.1	+4.2	
Ties	2–7	+ 2.4	+ 2.7	+ 3.8		+ 7.6*		+ 7.0		+ 6.4	−1.9	
	3–8	+ 3.0	+ 3.3	+ 4.8		+ 9.5*		+ 8.7		+ 8.0	−2.4	
	4–9	+ 3.8	+ 4.2	+ 6.0		+11.9*		+11.0		+10.1	−3.0	
Struts	2–6	− 1.7	− 1.9	− 2.7		− 5.4*		− 5.0		− 4.5	+1.4	
	3–7	− 2.5	− 2.8	− 4.0		− 7.9*		− 7.3		− 6.7	+2.0	
	4–8	− 3.4	− 3.8	− 5.4		−10.7*		− 9.9		− 9.1	+2.7	

slightly lower than the stress (+14.4) from dead and snow loads combined.

The next column contains the stresses on the basis of assumption (*a*) in the preceding article by which a uniform vertical load of 30 lbs. per square foot of horizontal projection over the whole roof is assumed as equivalent in its effect to the wind and snow loads combined. This load must be increased by 18 lbs. for the dead load, making the total load 48 lbs. per square foot. The corresponding stresses are found by multiplying the dead-load stresses by the ratio $48/18 = 2\frac{2}{3}$. For the chord members these stresses agree substantially with the maximum combined stresses and they are usually slightly greater; but for the web members they are uniformly about one-sixth lower.

General Deductions. The three assumptions governing the combination of wind and snow loads are indicated in Fig. 21. The loads distributed over each half-truss are represented by single arrows on which the loads per square foot are shown. The fixed and roller ends of the trusses are distinguished by *F* and *R* respectively.



Max. Wind + ½ Snow
(*a*)

Max. Snow + ½ Wind.
(*b*)

"Equivalent" Load.
(*c*)

FIG. 21.

The wind-load stresses in Table II are the maximum stresses—namely, the stresses in the fixed-ended half of the truss when the wind acts on that side of the roof, as in Figs. (*a*) and (*b*). It has been seen, however, that the maximum stresses in the lower-chord members in the roller-ended half of the truss are developed when the wind acts on that side, but that these stresses are less than those given in the table for corresponding members in the fixed-ended side by the magnitude of the horizontal component, $H = 4.3$, of the resultant wind load ΣW. In practice no account is taken, however, of this slight reduction of stress in proportioning the lower-chord members in the roller-ended half of the truss.

The web stresses in the left half of the truss were found to be greatest for the conditions shown in Fig. (*a*) and least for those in Fig. (*c*). It is instructive to observe in that connection that the stress in a web member is affected only by the loads between the *nearer* support and a section cutting that member and two others. The reaction itself cannot affect the stress, since it acts at the moment center of the web member. A load normal to the rafter produces a greater stress in the web members than an equal vertical load applied at the same point, because its lever arm about the moment center of these members—i.e., the end joint—is measured along the rafters instead of horizontally. From these considerations it might have been anticipated that the web stresses would be greatest in Fig. (*a*) and least in Fig. (*c*).

The stresses under the heading " wind uplift " are based on an assumed negative pressure on the leeward side one-half as great as the assumed positive pressure on the windward side. These values are therefore opposite in sign and one-half as great as those under the heading " wind load." They are also seen to be opposite in sign to the dead-load stresses but smaller in magnitude throughout, so that the resultant stresses are in no case reversed. In the design of the truss members under these conditions no account need therefore be taken of the stresses from negative wind pressures, but, as previously stated, it is desirable to avoid the use of flexible members incapable of resisting compressive stresses.

The following conclusions from the foregoing analysis with respect to wind stresses are applicable in general to roof trusses with straight rafters and horizontal lower chords meeting the rafters at the supports:

1. The web stresses in the leeward half are zero.
2. The lower-chord stresses are greater in the windward than in the leeward half, and greatest when the wind acts on the fixed-ended side of a truss supported on rollers at the other end, as in Figs. (*a*) and (*b*).
3. The upper-chord stresses are unaffected by the conditions at the supports, and their value on the leeward side is constant.

In the truss here considered the stresses in the upper chord on the windward side exceeded the stress on the leeward side. The relation between these stresses depends, however, on the slope of the rafters and the character of the web system. The stress on the leeward side sometimes governs for certain upper-chord members. Thus if the diagonals are inclined in the opposite direction, as in Fig. 22—i.e., if the truss is of the Howe type, the wind-load stress in member 8–9 is less than in the leeward rafter 1′–9; for considering moments about 5 of the forces to the right of sections q and u in turn, stress $1'\text{-}9 = R' \times \frac{1}{2}l/z$ and stress $8\text{-}9 = (R' \times \frac{1}{2}l - \frac{1}{2}Wv)/z$.

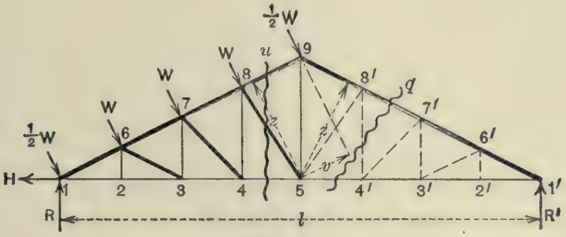


FIG. 22.

Again, if the rafters are inclined at angles of 60° to the horizontal, Fig. 23, the resultant, ΣW, of the wind loads will pass through the leeward support and the reaction at the windward support will be horizontal, if that support is fixed, or zero if it rests on rollers. In either case the stress in the lowest member, 1–4, of the windward rafter is tensile, as may be seen by taking moments of the forces to the left of a section cutting 1–4 and 1–2 about 2. The stress in 4–5 is compressive as may be seen by constructing a stress diagram, and it is much smaller than the compressive stress in the leeward rafter 1–5. If the joint 3 is located at the intersection of the resultant ΣW_1, of the loads at 1 and 4, Fig. (*b*), with the lower chord, stress 4–5 is zero, as is evident by taking moments of the forces to the left of section q about 3,

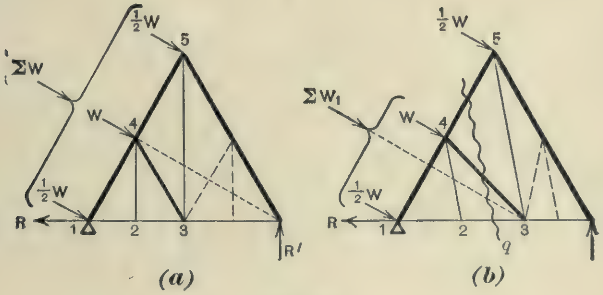


FIG. 23.

the moment center of 4–5. In this case both R and ΣW_1 intersect 3, and therefore their resultant passes through that point. Finally, if in Figs. (a) and (b) the right support is fixed and the left support is on rollers, the reaction at the latter is zero, and the entire lower chord is then in compression.

72. Reversed Stresses. Maximum and Minimum Stresses. In the truss analyzed in the preceding article the stresses in any given member from the dead, snow and wind loads are of the same character. The stresses in roof trusses in general from snow loads are obviously of the same character as those from dead loads if the snow is assumed to cover the entire truss. This is usually true also if the snow load is assumed as symmetrically distributed on roofs of varying inclinations, even though certain parts of the roof surface are regarded as free from snow on account of their steepness.

The wind-load stresses in some members may be opposite in character to those due to the dead load, especially in the case of trusses supported on columns stayed by knee-braces (Art. 75) and in three-hinged arches. If such a wind-load stress exceeds the dead-load stress in a given member the resultant stress from wind and dead loads combined is opposite in character to the dead-load stress, and the member is then subject to ‘ reversed ’ stress and must be designed accordingly by methods which will be considered in Vol. III.

If the dead-load stress, D, in a given member is subject to partial or complete reversal from the wind-load stress, W, then $D-W$ is called the ‘ minimum stress ’ in the member. If $W>D$, the minimum stress is opposite in sign to the dead-load stress. In roof trusses the dead-load stress is usually the minimum stress. The greatest resultant stress from the dead, snow and wind loads combined is the ‘ maximum stress ’ in the member. The minimum stress may be defined as the stress which is algebraically farthest from the maximum stress, this algebraic difference being the range of stress to which the member is subject. In the design of bridge members the allowable working stress is sometimes determined as a function of this range of stress, although in the design of roof-truss members this method is seldom or never used. It will be shown later that this method is not rational, even in its application to bridge members, and it will not be considered further in the treatment of roof trusses.

73. Roof Trusses with Chords of Varying Inclinations. The lower chord of a roof truss is frequently inclined (Fig. 24) either to produce a desired architectural effect, or to gain overhead clearance without the added cost of increasing the heights of the walls correspondingly. The saving in the walls will usually be partially offset by the increased weight of the trusses, although the latter may be largely or wholly neutralized by increasing the slope of the rafters.

Conditions for Maximum Wind-load Stresses. It has been seen (Art. 71) that the vertical components of the reactions are not affected by the conditions at the supports. To investigate the influence of the end conditions on the wind-load stresses in the trusses to be now partly analyzed it is only necessary therefore to consider the characters of the stresses produced by a possible horizontal reaction H at the leeward support, acting in the direction shown in Figs. (a) and (b), as compared with the characters of the stresses produced by the other forces to the leeward side of a section through the member in question. If these two stresses for any given member are opposite in character the magnitude of the stress produced by H need be considered only for members in which it is liable to exceed the other stress. Even in such cases the reversed wind-load stress affects the design of the member only if it exceeds the dead-load stress, which is opposite in character for ordinary roof trusses.

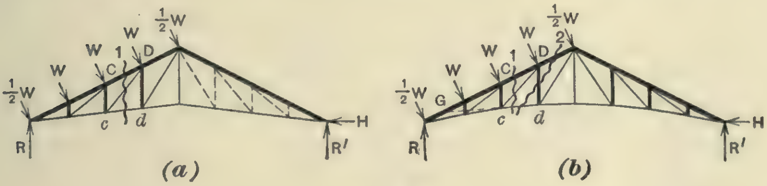


FIG. 24.

Upper Chord. Considering any upper-chord member as, for example, CD, Figs. (a) and (b), whose moment center lies at c, the moment of H about that point is clockwise and the stress in that member from H is therefore tensile. In roof trusses of the usual types and proportions the upper-chord stresses produced by the other forces, R' and ΣW, to the right of any section are compressive, and much greater than the tensile stress from

H. It may be concluded therefore that the wind-load stresses in the upper chord are maximum when $H = 0$—i.e., when the wind acts on the fixed-ended side of the roof, and when the other end rests on rollers. Again, the upper-chord stresses on the windward side of the truss are in nearly all cases greater than the stresses in corresponding members on the leeward side, although, as shown for trusses with horizontal lower chords in Art. 71 (General Deductions), that is not invariably true.

Lower Chord. Selecting any lower chord member as, for example cd, Figs. (a) and (b), it is seen by moments about D, that the stress from H is compressive and therefore opposite in character to the stress from the other forces, R' and ΣW, to the right of the section, while in all ordinary cases the latter stress is much greater. The conclusions are therefore precisely the same as stated above for the upper-chord stresses, except that for the lower chord it seldom or never happens that the stresses on the windward side are exceeded by the stresses in corresponding members on the leeward side.

Web Members. If the slope of the lower chord is constant from the end to the center of the truss, as in Fig. (a), the moment center of every web member lies at the end joint. Since the moment of H about that moment center is zero, the web stresses are in this case unaffected by the conditions at the supports—i.e., the web stresses on the fixed-ended side, when the wind acts on that side, are equal to the stresses on the roller-ended side when the wind acts on that side.

In Fig. (b) the slope of the lower chord varies for successive panels. In this case when the wind acts as shown the stress is zero only in the last vertical to the right. If the chord is horizontal for the two middle-panels, the stress in the middle vertical is also zero.

Taking sections 1 and 2 the moment center of both Dc and Dd lies at G, the intersection of the upper chord with cd produced. By moments about G it is seen that H produces compression in Dc and tension in Dd. As in the case of the chords, these web stresses from H are under usual conditions opposite in character and smaller than those due to the other forces, R' and ΣW, to the right of any section that may be taken on the windward side. Hence the same conclusions apply.

Thus, *the wind-load stresses in trusses with straight upper* *chords and concave (downward) lower chords meeting each other at the supports, are in general maximum in the fixed-ended half of the truss, when the wind acts on that side of the roof. When the truss is fixed at both ends, the stresses are maximum in the windward half.*

For practical reasons no reduction is made in the sizes of the members in the roller-ended half of a truss because of the somewhat smaller maximum stresses in these members than in corresponding members in the fixed-ended half.

Trusses of Other Types. As regards chord stresses the above conclusions apply also to trusses in which the straight upper chords do not meet the curved lower chords (Fig. 12*b*), and to crescent trusses (Fig. 13) in which both chords are curved. In trusses of the former type the maximum web stresses are also developed under the conditions above stated, but the web members near the middle of each half are subject to reversed stresses when the wind acts on the other half, and these stresses must be duly investigated (Art. 74). With respect to the web stresses in trusses in which both chords are curved no broad generalizations are possible. The stress diagrams must be drawn for the wind acting on each side in turn. The resulting web stresses must be combined with the dead and snow-load stresses for maximum stress, and with the dead-load stress for reversed stresses. Under ordinary conditions the chord stresses are not subject to reversal in trusses of any of these types.

In constructing the wind-load stress diagrams for trusses in which the web members are active in both halves of the truss it is desirable to minimize the effect of cumulative errors by working first from one end, and then from the other end, to the center joints, at which the diagrams should close. The stress in a chord member at or adjacent to the center may be computed by the method of moments (lever arms calculated or scaled) and compared with the values of the corresponding stress found graphically, so that in case of serious disagreement either or both segments of the diagram may be redrawn. An illustrative example of this procedure will be given later (Art. 75).

74. Counters. The allowable working stresses in tension are greater than in compression. Moreover, the former are not influenced by the length of the member, whereas the latter decrease rapidly as the slenderness ratio (length of member divided by least radius of gyration of cross-section) increases. It is economical, therefore, to design a truss so that the longer web members will act in tension for the principal loading or, if possible, for all loading. Hence, in the case of a steel roof truss, it is preferable that the diagonals rather than the verticals should act in tension for dead and snow loads. For wind loads, the stresses in the diagonals may be reversed, in which case the member may be counterbraced (Art. 48), or a second tension diagonal, sloping in the opposite direction, may be introduced. Such diagonals, known as *counters*, are represented by the broken lines in Fig. 25. *The stresses in the counters from dead load, or from any symmetrical loading on the truss, are zero.* The counters are often rods whose length may be adjusted by means of 'turnbuckles,' 'sleeve-nuts,' or other devices. Under ideal conditions the counter is assumed to be adjusted so as to hang slack until the tension in the main diagonal in the same panel is exhausted by loads which produce compression in that member. As the tension is thus reduced the main diagonal shortens, which causes the counter to lengthen. The latter is not assumed to come into play, however, until the tension in the former is completely neutralized. Practically, it is not possible nor desirable that these conditions should be strictly met. If adjustable members are used for the counters they are screwed up to 'initial tension' to prevent vibration under suddenly applied loading. If plates or rolled 'shapes' are used, they are made to exact length, the end connections are riveted, and no provision is made for adjustment. In this case also, some initial stress may be introduced by inaccuracy of fit. It will be assumed at first that the main diagonal and the counter cannot act simultaneously, and the effect of initial stresses will be afterward considered.

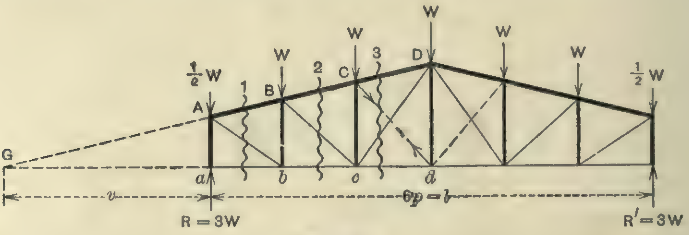


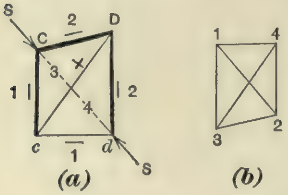
FIG. 25.

Referring to Fig. 25, let W denote the dead apex loads. The main diagonals must be so disposed that they will act in tension under dead loading. Taking sections 1, 2 and 3 in turn, and considering the moments, M_1, M_2 and M_3 of the external forces to the left of these sections about the moment center G, for a truss of the proportions shown (1) M_1 and M_2 act counterclockwise, whence Ab and Bc are in tension, and (2) M_3 acts clockwise, whence Dc is in tension. If the slope of the roof is increased, v, the lever arm of R, is decreased. When v reaches such a value that $M_2 = (3W - \frac{1}{2}W) \; - W(v+p) = 0$, stress Bc becomes zero. If v is decreased further, stress Bc becomes compressive. In that case Bc should be replaced by Cb, since the latter will then act in tension under dead loading.

Let it be assumed now that the left end of the truss rests on rollers, and that the right half of the roof is subjected to wind pressure normal to its surface. The reaction R is then vertical, and it is the only external force on the left half of the truss due to wind loading. Taking moments about G as before, the wind-load stresses in Ab and Bc are tensile, but the stress in Dc is now compressive, and therefore tends to neutralize the dead-load tensile stress in that member. If the wind-load stress in Dc exceeds the dead-load stress, either Dc must be counterbraced or—what is usually more economical—counter Cd must be inserted. The stress in the counter is then the algebraic sum of the dead and wind-load stresses in that member on the assumption that Dc does not exist. This change in the truss affects also the stresses in chord members CD and cd, and in the verticals Cc and Dd. The corresponding changes in the dead-load stress diagram, for Cd instead of Dc in action, are readily made. It is important to observe that *these modified dead-load stresses must be used in combining dead and wind-load stresses.* Similarly, in combining dead, wind and snow-load stresses, it must be ascertained first whether the main diagonal or the counter is in action for this combined loading. If the counter acts, the modified dead and snow-load stresses must be used. Several examples in illustration of the application of this principle to bridge trusses will be given hereafter (Arts. 103–105).

Initial Stresses. Let it be assumed that counter Cd, Fig. 25, is given initial tension by tightening the turnbuckles when the truss is subject only to dead-load stress. This is equivalent to applying two equal and opposite forces at C and d acting in the directions indicated by arrows. These forces have evidently no effect on the reactions and therefore the stresses to either side of the panel in question are also unaffected. The effect of the initial stress, S, is thus seen to be confined wholly to the quadrangular frame shown separately in Fig. 26, and the stresses thus induced may be found graphically as in Fig. (*b*). The stress in the main diagonal cD is tensile and the other four stresses are compressive. If CD were horizontal, the tensile stresses in both diagonals would be equal.

It is sometimes erroneously assumed that such initial stresses should be combined with the stresses due to loads on the truss. But, unless the initial stress is very excessive, through improper adjustment of the counter, the tensile stress in either diagonal will be wholly neutralized when the loading is such as to produce maximum tension in the other. For as the tension in one diagonal is increased, the length of that diagonal is increased, whereas the length of the other diagonal, and therefore its stress, is decreased. The diagonal in which the stress is exhausted may be assumed to be removed from the truss, and the stresses in all other members are then the same as though no initial stresses had originally existed.

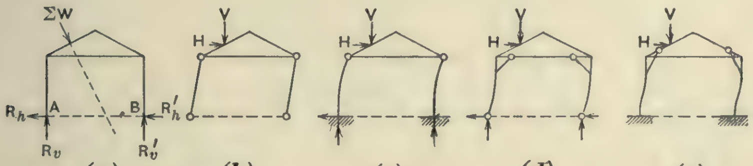


(*a*) (*b*)

FIG. 26.

A clear understanding of the action of counters is especially important in the analysis of the stresses in bridge trusses, and the subject will receive further attention in that connection (Art. 81).

75. Wind-load Stresses in Trusses Supported on Columns. Let ΣW, Fig. 27 (*a*), denote the resultant of the wind pressure on a roof supported on columns. If the line of action of this force lies between A and B, the vertical components, R_v and R_v', of the reactions are both upward, and their values may be found by moments about B and A respectively. The horizontal components are statically indeterminate (Art. 23), and are usually assumed to be equal. Thus if H and V denote respectively the horizontal and vertical components of ΣW, $R_h = R_h' = \frac{1}{2}H$. The horizontal component of the wind pressure tends to produce the distortion shown in Fig. (*b*), and if the columns are hinged top and bottom, the structure will collapse. To ensure stability the columns must be fixed-ended at the base, Fig. (*c*), at the top, Fig. (*d*), or at both extremities, Fig. (*e*). In order that a column may properly be regarded as fixed-ended, it must be capable of resisting the maximum bending moment which can be developed at that end by the wind pressure. Unless the 'anchorages' at the column bases are designed to meet this requirement, which is usually not the case, the bases should be assumed as hinged. It is not practicable in an economic sense to secure a condition of fixed-endedness at the top by connecting the column with the truss by means of a single riveted joint. For trusses in which the chords meet at a common point, as in Fig. 27, this result is best accomplished by means of knee-braces, which may be connected at their upper extremities with lower-chord joints, Fig. (*d*), or upper-chord joints, Fig. (*e*). In practice the condition represented in Fig. (*c*) seldom occurs. If the columns are fixed-ended at the base, as in this case, the bending moments in the columns may be greatly reduced by introducing knee-braces at the top, as in Fig. (*e*).



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 27.

The conditions shown in Figs. (*d*) and (*e*) will now be separately analyzed.

(*a*) *Columns Hinged at Base.* The truss shown in Fig. 28 is that previously analyzed in Art. 71. The line of action of the resultant of the wind pressure, 9.6 thousand pounds, is perpendicular to the rafter at its center. The vertical components of the reactions may be best found as the algebraic sums of the reactions from the vertical and horizontal components of the wind pressure, considered separately. Thus

$$R_v = 8.57 \times \tfrac{3}{4} - 4.29 \times 26.25/50 = 4.18,$$
$$R_v' = 8.57 - 4.18 = 4.39.$$

The horizontal reactions will be assumed equal, whence

$$R_h = R_h' = 4.29 \times \tfrac{1}{2} = 2.15.$$

In order that the effect of the knee-braces on the wind-load stresses in the truss may be judged by direct comparison of the stresses in this case with those found in Art. 71, it will be assumed that the sides of the building are open, or that the side walls are independently supported and that they transmit no wind pressure to the steel framing. If the conditions are otherwise, additional horizontal wind forces of suitable magnitudes must be added at C and B, and the reactions at the column bases modified accordingly.

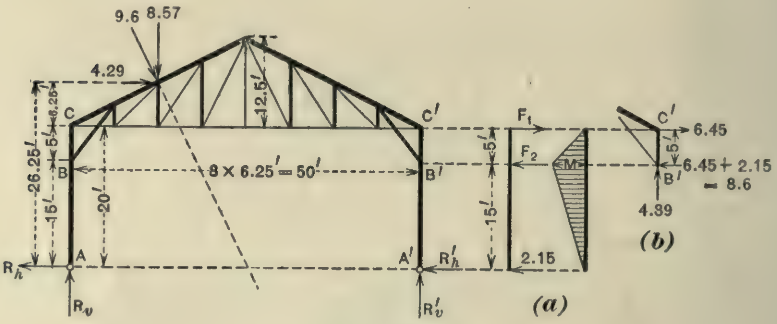


FIG. 28.

Columns. The horizontal reactions at A and A' will evidently produce bending in the columns and develop horizontal forces F_1 at C and C', and F_2 at B and B'. With respect to the truss and knee-brace these forces will act in the directions shown in Fig. (*a*), but with respect to the columns they will act in the opposite directions. Their magnitudes may be found by moments. Thus $F_1 = 2.15 \times 15/5 = 6.45$, and $F_2 = 2.15 \times 20/5 = 8.6$. Note that F_1 and F_2 are components of 2.15, and that 2.15 is therefore their resultant—not their equilibrant. These three forces, in the directions in which they are shown in Fig. (*a*), are therefore not in equilibrium. Considering the horizontal forces at B' and C', Fig. (*b*), it is seen that $R_h'(=2.15)$ has been transferred to B' through the development of the couple whose moment is $6.45 \times 5 = 2.15 \times 15$ (Art. 15).

The bending moment at any section of the column is represented by the horizontal intercept at the corresponding level in the shaded moment diagram, Fig. (*a*). The bending moments are maximum at B and B' where $M = 2.15 \times 15 = 32.3$ thousand foot-pounds. The columns must be proportioned for this bending moment combined with the maximum compression from dead, snow and wind loads. The compressive wind-load stresses in the lower parts of the columns, AB and $A'B'$, are R_v and R_v' respectively. This stress is therefore somewhat greater in the leeward column, where it equals 4.4. As will be seen presently, however, the greater stress from wind occurs in the upper part BC, of the windward column, where its value is 15.4 (Table III). From Art. 71 the panel concentrations from dead load are 1.7 and from snow load 1.9. The corresponding reactions at C and C' are $1.7 \times 4 = 6.8$ and $1.9 \times 4 = 7.6$.* Hence the maximum combined stresses in the columns are:

For Dead + Wind + $\frac{1}{2}$ Snow, $6.8 + 15.4 + \frac{1}{2} \times 7.6 = (-)26.0$;
For Dead + Snow + $\frac{1}{2}$ Wind, $6.8 + 7.6 + \frac{1}{2} \times 15.4 = (-)22.1$.

The columns must therefore be proportioned for a compressive stress of 26.0 thousand pounds combined with a bending moment of 32.3 thousand foot-pounds.

Truss and Knee-braces. The wind loads and the reactions at 1, 1′, 10 and 10′, Fig. 29, developed at these points by the forces at the column bases, are as shown in that figure.

In the stress diagram, Fig. (*a*), the force polygon *abcdefghikla* of the external forces is first laid off. To draw the stress diagram, begin at joint 10 and proceed by consecutive joints in the left half of the truss to joint 9. Then begin at joint 10′ and work toward joint 9. On reaching joint 3′ and drawing $r's'$ the point s' should lie vertically above the point t' previously established in the analysis of joint 9. Passing then to joint 8′, $s't'$

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* Strictly speaking the vertical loads will be transmitted partly to B and B' through the knee-braces. The proportions of the loads thus transmitted depend on the stiffness of the truss, the column and the knee-braces. The reactions at C and C' are therefore statically indeterminate. It is sufficiently accurate, however, to assume that vertical loads produce no stresses in the knee-braces.

is the closing line of the diagram. If s' should not be found to lie vertically above t', the points u and u' will not coincide, which would indicate that the stresses iu and iu' in members 4–5 and 4′–5 respectively, are not equal as they should be. In that case the true value of this stress may be found by moments and compared with the values found graphically, thereby determining whether one or both segments of the diagram should be redrawn. Thus taking section z and considering the moments of the forces to the right about 9, it is more convenient to use the two forces, R_h' and R_v', in Fig. 28 than the three forces in Fig. 29, whence

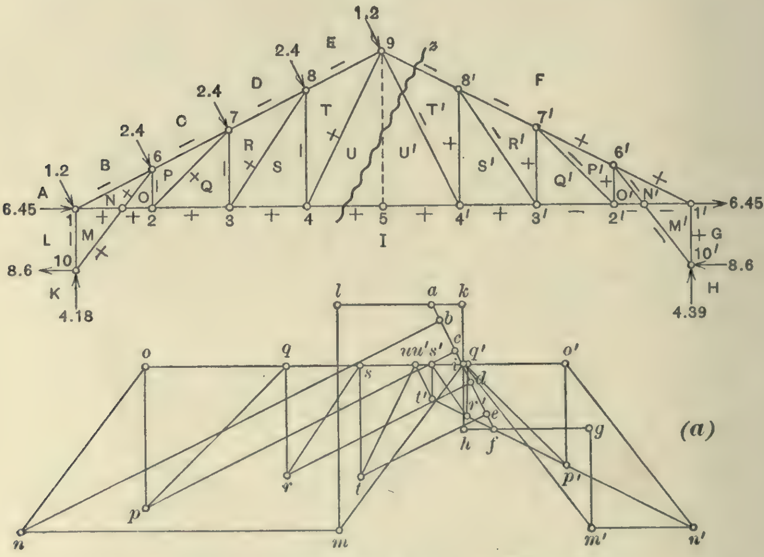


FIG. 29.

$$\text{Stress } iu = iu' = (4.39\times25 - 2.15\times32.5)\div12.5 = (+)3.19.$$

Graphically the value of the stress was found to be 3.2. The original truss diagram was drawn to a scale of 5 ft. to the inch, and the stress diagram to a scale of 3000 lbs. to the inch. The values of the stresses from Fig. (*a*) are shown in Table III.

(*b*) *Columns Fixed at Base.* This condition is shown in Fig. 27 (*e*) and to an enlarged scale for the leeward column in

Fig. 30 (*a*), in which the deformations are much exaggerated. Assuming joint C' as hinged, the curvature of the column will be of the general character shown in the figure, the tangent to the curve at B', making a small angle α with the vertical. Joint C' is commonly riveted, and it is sufficiently accurate for practical purposes to disregard the lateral deflection in $C'B'$ and to assume the column as strictly fixed-ended at B' as well as A'—i.e., to regard the tangents to the curve at these points as vertical, as in Fig. (*b*). Locating O' midway between A' and B' and drawing a vertical axis through this point, it is evident that for the usual condition of a column of constant cross-section, the curvature is symmetrical about this axis in the sense that the abscissas for equal ordinates above and below O' will be equal but opposite in sign—that is to say, proceeding downward, the curvature changes at O' from convex to concave to the right. This indicates that the bending moments above and below O' are opposite in character and that the bending moment at O' is zero. This is also apparent because, from the conditions of symmetry, the bending moments, M, at A' and B' are equal and opposite. The section of zero bending moment must therefore be at O', as shown in the shaded moment diagram. Since the bending moment at O' is zero, the column may be regarded as hinged at this point—i.e., capable of developing shear and axial stress, but no bending moment.

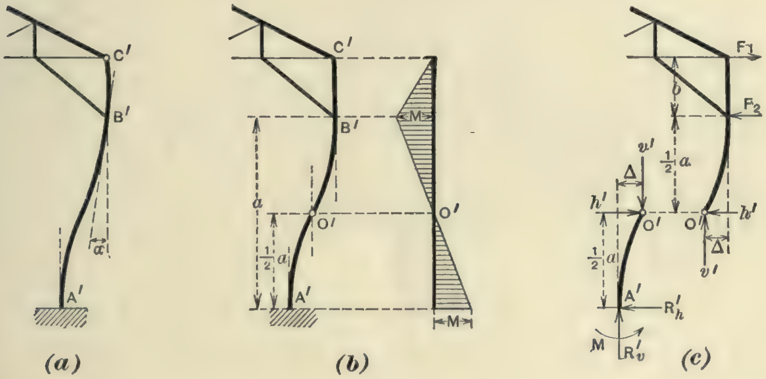


FIG. 30.

The lower part, $A'O'$, of the column is held in equilibrium by the forces shown in Fig. (*c*), in which $v' = R_v'$, $h' = R'_h$ and $M = h' \times \frac{1}{2}a$. Since the lateral deflection, Δ, is exceedingly small, the existence of the unbalanced moment, $v' \times \Delta$, which, strictly speaking, should be added to M, may be neglected without sensible error. With respect to the part of the column above O', the forces v' and h' evidently act in the opposite direction, as shown in the figure.

The forces which hold that part of the framework above a horizontal plane through OO' in equilibrium are completely shown in Fig. 31. The reactions at O and O' are precisely the same as for a roof supported on columns hinged at the bottom, whose height is $O'C' = \frac{1}{2}a + b$. It will be assumed, as before, that $h = h' = \frac{1}{2}H$, but the bending moments produced at A, A', C and C' by these forces are now only one-half as great as those at C and C' in the case of columns hinged at A and A'. The vertical components, v and v', of the reactions may be found by moments about O' and O, respectively; or one of these forces may be thus found, and the other by subtracting this value from V.

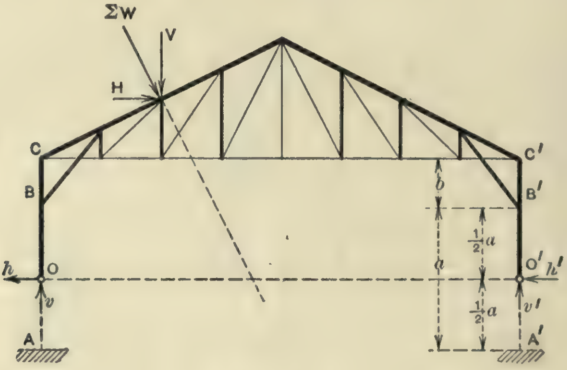


FIG. 31.

It has been seen that v, v', h and h' represent also the corresponding components of the reactions at the column bases A and A'.

Resuming the numeric data used in Case (*a*), Fig. 28, the vertical reactions from the horizontal component, 4.29, of the load are now found by taking moments about points 7.5 ft. above A and A', corresponding to O and O' in Fig. 31. Thus

$$v = R_v = 8.57 \times \tfrac{3}{4} - 4.29 \times 18.75/50 = 4.82;$$

$$v' = R_v' = 8.57 - 4.82 = 3.75,$$

and, as before,

$$h = h' = R_h = R_h' = 2.15.$$

Columns. In Fig. 30 (c), $\frac{1}{2}a = 7.5$ ft. and $b = 5$ ft. Hence $F_1 = 2.15 \times 7.5/5 = 3.23$ and $F_2 = 3.23 + 2.15 = 5.38$, or, by moments $F_2 = 2.15 \times 12.5/5 = 5.38$. The maximum bending moments at B and B' are now $M = 2.15 \times 7.5 = 16.1$ thousand foot-pounds. The columns must be proportioned for this bending moment combined with the maximum compressive stress from dead, snow and wind loads. As in Case (a), the greatest compressive stress from wind occurs in the upper part, BC, of the windward column, where its value is 11.9. The dead and snow-load stresses are the same as in Case (a). Thus the maximum combined stresses in the columns are:

For Dead + Wind + ½ Snow, $6.8 + 11.9 + \frac{1}{2} \times 7.6 = (-)\ 22.5$;

For Dead + Snow + ½ Wind, $6.8 + 7.6 + \frac{1}{2} \times 11.9 = (-)\ 20.4$.

The columns must therefore be proportioned for a compressive stress of 22.5 thousand pounds combined with a bending moment of 16.1 thousand foot-pounds.

Truss and Knee-braces. The forces induced in this case at 1, 1′, 10 and 10′, Fig. 29, are as above computed, and the stress diagram (not shown) may be constructed as explained in connection with that figure. The resulting wind stresses are given in Table III.

The marked increase in the wind-load stresses in the case of a truss supported on columns stayed transversely by knee-braces, as compared with a fixed-ended truss resting on walls, is apparent from the values in Table III. The stresses for the windward side of the fixed-ended truss are taken from Table II, Art. 71, and those for the leeward side from Fig. 19. The dead-load stresses in that table are exceeded in certain members by the wind-load stresses opposite in character, distinguished by asterisks in Table III. This again emphasizes the importance of avoiding the use of flexible tension members incapable of developing at least moderate compressive stresses. The increase n the stresses in trusses with knee-braces is relatively greater for the web members than for the chord members, and greater for hinged-ended than for fixed-ended columns, as was to be anticipated. The stresses on the leeward side are seen to be smaller in all cases than the stresses in corresponding members on the windward side, and usually opposite in character.

TABLE III

WIND-LOAD STRESSES

Position.	Windward Side.				Leeward Side.			
	Member.	Both Ends Fixed.	With Knee-braces.		Member.	Both Ends Fixed.	With Knee-braces.	
			Columns Hinged at Base.	Columns Fixed at Base.			Columns Hinged at Base.	Columns Fixed at Base.
Lower Chord	1–2	+11.3	+21.7	+17.9	1′–2′	+3.3	− 7.0	−3.1
	2–3	+ 8.7	+12.1	+10.8	2′–3′	+3.3	− 0.1	+1.1
	3–4	+ 6.0	+ 7.1	+ 6.8	3′–4′	+3.3	+ 2.1	+2.6
	4–5	+ 3.3	+ 3.2	+ 3.4	4′–5	+3.3	+ 3.2	+3.4
Upper Chord	1–6	−10.8	−32.0	−24.2	1′–6′	−6.0	+15.1*	+7.1
	6–7	−12.0	−23.7	−19.4	6′–7′	−6.0	+ 5.4	+1.1
	7–8	−10.2	−14.1	−12.7	7′–8′	−6.0	− 2.2	−3.7
	8–9	− 8.4	− 9.8	− 9.4	8′–9	−6.0	− 4.8	−5.3
Ties	2–7	+ 3.8	+13.6	+10.0	2′–7′	0	− 9.7*	−6.0*
	3–8	+ 4.8	+ 9.0	+ 7.4	3′–8′	0	− 4.1*	−2.5
	4–9	+ 6.0	+ 8.6	+ 7.6	4′–9	0	− 2.5	−1.6
Struts	2–6	− 2.7	− 9.6	− 7.0	2′–6′	0	+ 6.8*	+4.2*
	3–7	− 4.0	− 7.5	− 6.2	3′–7′	0	+ 3.4*	+2.1
	4–8	− 5.4	− 7.7	− 6.8	4′–8′	0	+ 2.3	+1.4
Knee-Brace	—	——	+14.1	+ 8.8	—	—	−14.1	−8.8
Column	1–10	——	−15.4	−11.9	1′–10′	—	+ 6.7	+3.2

* Stresses opposite in character to and greater than dead-load stresses.

The stresses in the trusses with knee-braces would be much greater than the values in Table III if the wind load on the vertical sides of the building had been assumed to be transmitted to the columns, as would occur, for example, if the sides were enclosed with corrugated steel sheets fastened to horizontal members, called 'girts,' attached at intervals to the columns. If the side walls consist of brick, concrete or other material the proportion of the wind pressure transmitted to the columns must be judged according to the circumstances of the case.

Trusses of the type shown in Fig. 12, Art. 63, in which the upper and lower chords do not meet at a common point, do not require knee-braces when supported on columns. The latter are in that case extended to the top chord. The horizontal forces at the bases of the columns will develop horizontal reactions at the points of junction of the columns with both chords, which are found in the manner explained for knee-braces.

76. Formulas for Weight of Roof Trusses. The weight of steel roof trusses may be estimated approximately by the data given in Art. 65. It is stated in that article that the total weight of trusses, bracing and purlins for roofs of the same span may easily vary 25 per cent or more from circumstances of which no simple formula can take proper account. In formulas for the weight of roof trusses, the span length is usually the only variable. Such formulas are unreliable unless applied under conditions similar, in every important respect, to those used in their derivation. These conditions are, however, seldom adequately stated. The weights obtained from some of the most widely quoted formulas exhibit in some cases differences of over 100 per cent.

The weight of a steel roof truss of a given span is more or less seriously affected by each of the following elements:

1. Type of truss and slope of rafters.
2. Length of bays.
3. Assumed magnitudes and distribution of loads.
4. Assumed combination of loads.
5. Increased wind stresses in trusses supported on columns, as compared with trusses resting on walls (Art. 75).
6. Assumed working stresses.
7. Bending stresses in rafters supporting purlins between panel points.
8. Minimum allowable thickness of metal.

The minimum allowable thickness of metal in roof trusses should preferably be $\frac{5}{16}$ in., and never less than $\frac{1}{4}$ in., although $\frac{3}{16}$-in. metal has frequently been used.

For reasons above stated no formula for the weight of roof trusses will be derived or quoted.

77. Three-hinged Arch. It has been seen (Art. 26) that the reactions of an arch are inclined forces, for both vertical and horizontal loads. The horizontal components of the reactions must balance the so-called 'thrust' of the arch. From Fig. 10, Art. 26, it is seen that for vertical loads the thrust at both supports is equal and is exerted outward, i.e., the horizontal components of the reactions must act inward. Instead of providing heavy buttresses or anchorages in massive piers to develop these horizontal forces it is more economical to connect the supports of the arch by a horizontal tie, and to place one end on rollers, as in Fig. 32. Under vertical loads the stress, $S(=R_h=R_h')$, in the tie is evidently tensile. The arch, inclusive of the tie, may now be regarded as a truss, the vertical loads being held in equilibrium by the vertical reactions R_v and R_v', whose magnitudes may be found, as for a truss, by moments about B and A respectively. To find the stress in the tie, take a section cutting the central hinge C, and the tie. If M_C denotes the resultant moment of the vertical reaction and loads to either side of the section about C, $S=M_C/h$. The stress in the tie may also be found by Eq. (7), Art. 26.

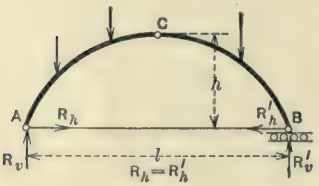


Fig. 32.

The introduction of a tie and rollers also tends to minimize the vertical movement at the crown C, under variations of length in the arch members from changes in temperature. Thus if the horizontal distance, l, from A to B remains unchanged, C will rise under rising temperatures. A lengthening of l will cause a fall at C. Hence the lengthening of the tie under rising temperatures tends to counteract the rise at C from the lengthening of the arch members.

From Fig. 11, Art. 26, it is seen that for horizontal loading on one side of the arch, the horizontal component, R_h', of the reaction at the *farther* support acts *inward*, as for vertical loads;

but that the horizontal component, R_h, of the reaction at the *nearer* support acts *outward.* If a horizontal member AB is provided, and one of the supports is mounted on rollers, as in Fig. 33, the arch, inclusive of member AB, may be regarded as a truss. In Fig. (*a*) the horizontal load W acts on the fixed-ended half, and in Fig. (*b*) on the roller-ended half of the structure. In either case the horizontal component of the reaction at the fixed end is W, its direction being opposite to that of the load W, as shown in the figure. This horizontal force W must be developed by the anchorage at the fixed end.

The stress S in member AB may be either tensile or compressive. This stress may be found by applying $\Sigma H = 0$ at the roller end, and considering that S must equal the horizontal component of the reaction at that end for the arch proper. Thus in Fig. (*a*), from the forces at B, $S = +R_h'$, and in Fig. (*b*), from the forces at A, $S = -R_h$, in which R_h' and R_h may be found from Eqs. (9) and (10), Art. 26.

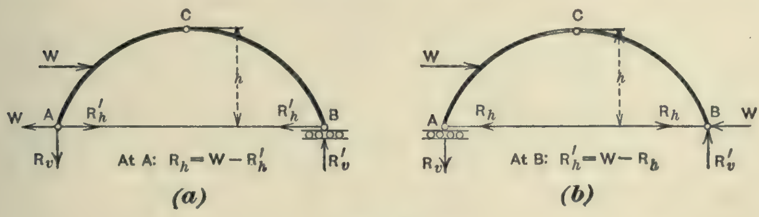


FIG. 33.

The resultant of the horizontal forces at the fixed end is $W - R_h'$ in Fig. (*a*), and $W - R_h$ in Fig. (*b*). These resultants are the horizontal components of the reactions which would be developed at the corresponding supports of the arch proper if the member AB did not exist.

The values of the vertical forces R_v and R_v' are not affected by the introduction of member AB, as is evident by considering that these forces are direct functions of the moment of W about B and A respectively.

In the common case of inclined forces the inclined reactions may be found, as in Art. 26 (*d*), on the assumption that the member AB and the rollers do not exist. The stress in AB is then the horizontal component of the reaction at the roller end.

In general, the tensile stress in AB from dead load is not subject to reversal by the compressive stress from the horizontal components of the wind pressures on the roller-ended side. Member AB is therefore usually designed as a flexible tension member incapable of developing compressive stresses, and it is placed below the floor level, where its presence is unobjectionable.

After the reactions are determined, the stresses in a three-hinged arch may be found graphically in the usual way by drawing force polygons, first, of the external forces, and then of the forces at each joint in turn. For reasons stated in Art. 73 (last paragraph) it is advisable, in the case of unsymmetrical loading, to proceed successively from each end joint toward the center.

Note.—The aim has been to present all important principles in the analysis of ordinary roof trusses in this chapter and in Chapter V., Arts. 52–55, inclusive. The critical treatment of the Pratt truss (Arts. 71 and 75) on the basis of numeric data, is believed to be far more advantageous to the student than the multiplication of stress diagrams, involving no new principles, for various types of trusses. For reasons stated in Art. 61, graphic methods are, in general, best adapted to the determination of the stresses in roof trusses. With such exceptions as have been noted, the reactions may usually be found more conveniently by numeric methods. The construction of stress diagrams for ordinary roof trusses is a simple process in which a fair degree of proficiency may be readily acquired by a little individual practice, but in no other way.

CHAPTER VII

STRESSES IN BRIDGE TRUSSES WITH HORIZONTAL CHORDS

A. General Considerations

78. Classification of Bridges. Bridges are classified as 'railway bridges' or 'highway bridges,' according to the general character of the service which they are designed to meet.

To ensure lateral stability under wind pressure, and under the action of centrifugal forces if the rails are laid on a curve, the main trusses of a bridge are connected by *lateral bracing* in the plane of one or both chords. The bracing in the plane of the upper chord also serves to 'stay' that chord against bending laterally under direct compression.

A *through bridge* is one in which the floor system lies near the plane of the lower chords, and in which the upper chords are connected by lateral bracing. If the trusses are so low that such overhead bracing cannot be used, the top chords are stayed laterally by inclined struts, called 'knee-braces,' connecting that chord at each panel point with the transverse floor beam at the corresponding lower joint of the truss. Such a truss is called a *pony truss*.

A *half-through* bridge is one in which the floor system occupies a position intermediate between the planes of the upper and lower chords. In a bridge of this type there is seldom room for overhead bracing, so that the top chord is usually connected to the floor beams by some form of bracket or knee-brace, as in the case of a pony truss.

A *deck bridge* is one in which the floor system lies near the plane of the upper chords.

The chord in the vicinity of the floor system is called the *loaded* chord, and the other the *unloaded* chord, as a convenient verbal distinction, although in reality the loads are seldom applied directly to the chords, but are communicated to the trusses only at the joints to which they are transmitted by the floor system.

79. Assumed Distribution of Dead Load. A truss supported only at its extremities, and under such conditions as to render the reactions statically determinate (Art. 23), is called a *simple* truss.

In trusses with horizontal chords, the chord stresses vary directly as the bending moments, and the web stresses, if the panels are equal, directly as the shears. This may be seen for chord stresses from Eq. (9), Art. 57, by observing that the numerator denotes the bending moment at the moment center, and that h is constant throughout the truss; and for web stresses by reference to Art. 58. It has been seen in Chapter IV that the maximum bending moments in a simple beam or truss increase from the ends toward the center, and that the maximum shears vary in the opposite directions. It follows, therefore, that in simple trusses with horizontal chords, the weight of the chords increases from the ends toward the center, and that the weight of the web members increases in the opposite directions. This is true also of simple trusses with chords of moderate curvature. It may, therefore, be assumed in general with sufficient accuracy that the weight of simple trusses is uniformly distributed over the span. For the usual case of equal panels the weight of the floor system, i.e., floorbeams, stringers, track and flooring, may also be regarded as uniformly distributed, and that assumption may likewise be made, without much error, with respect to the lateral bracing.

The division of the dead load between the upper and lower chords is usually based on one of the following assumptions:

I. That the dead load is applied wholly to the loaded chord.

II. That two-thirds of the dead load is applied to the loaded chord, and one-third to the unloaded chord.

III. That the weight of the floor system is applied wholly to the loaded chord and that the weight of the trusses and lateral bracing is equally divided between both chords.

Assumption I is admissible only when the dead load is very small in relation to the live load as, for example, in the case of very short highway bridges.

Assumption II is needlessly inaccurate and is therefore not to be recommended.

Assumption III is probably as satisfactory as any general assumption that can be made. In the design of a bridge, the floor system is first proportioned and its weight computed. There can be no question that this weight should be assumed to be applied to the loaded chord, and the further assumption that the remaining dead load is equally divided between both chords is sufficiently accurate for practical purposes.

Let W_l = Dead apex load for loaded chord;

W_u = Dead apex load for unloaded chord;

p = panel length, in feet;

w_1 = weight of floor beams and stringers, per foot, per truss;

w_2 = weight of tracks and flooring, per foot, per truss;

w_3 = weight of truss and bracing, per foot, per truss.

Then

$$W_l = (w_1 + w_2 + \tfrac{1}{2}w_3)p;$$

$$W_u = \tfrac{1}{2}w_3p.$$

In the case of very large bridges the dead loads transmitted to the various panel points are sometimes carefully computed after the structure has been completely designed. The dead-load stresses are then revised, and the sections of the truss members modified accordingly.

80. Counterbalanced Loading. Equal loads symmetrically disposed with respect to the center line of a truss are said to be *counterbalanced.* If the loads on a truss are all counterbalanced the shear at a section through any panel is equal to the sum of the loads between that section and the center of the truss. If a load exists at the center of the truss, as in the case of a truss with an even number of panels, one-half of that load must be included in the sum.

Thus in Fig. 1, if the panels are equal the dead apex loads W_1 and W' are counterbalanced. If $W = W' + W_1$, the shear in panel *cd*, for example, is equal to $R(=3.5W) - 2W = 1.5W$, which is evidently equal to the loads between section 1 and the center of the truss, including one-half the loads at the center line.

The principal elements of a truss have been defined in Art. 48. In a truss in which the end web member is inclined and in compression, as in Fig. 1, that member is called an *inclined end post* or 'batter brace.' The joint B at the junction of this member and the top chord is called a 'hip joint.' In a truss of the type here shown the member Bb is called a *hip vertical*, 'end suspender' or 'end hanger.' From $\Sigma V=0$, applied to the forces at b, it is seen that the stress in the hip vertical is tensile, and equal to the apex load at b.

In a truss of this form the diagonals are designed as flexible members capable of acting in tension, but not in compression. Under dead load or counterbalanced loading in general, the stresses in the diagonals represented by broken lines are zero. These members are *counters* (Art. 74), and they act in tension when the live load occupies certain positions. The other diagonals are called *main diagonals* or 'main ties.'

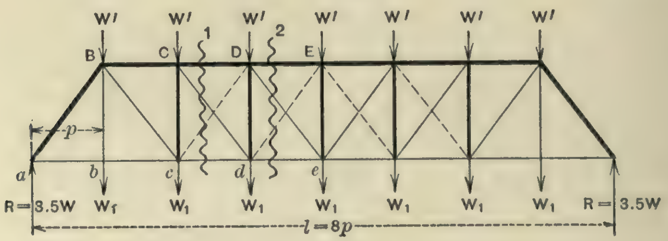


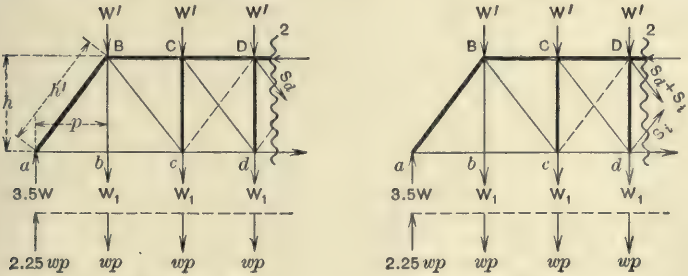
FIG. 1.

81. Counters. The action of counters in connection with roof trusses has already been considered (Art. 74). Referring to Fig. 1, Art. 80, that segment of the truss to the left of section 2 is shown in Fig. 2 (*a*), in which $W=W_1+W'$, as before. From $\Sigma V=0$, applied to the forces which hold this segment in equilibrium, it is seen that the vertical component, $3.5W-3(W_1+W')=0.5W$, of the dead-load stress, S_d, in the main diagonal De acts downward, and that this stress is therefore tensile. From the dimensions of the truss, $S_d=+0.5W\times h'/h$. In the absence of initial stress (Art. 74), the stress in the counter dE under the action of the dead load is zero.

Let it be assumed now that live apex loads, wp, are applied at joints b, c and d, only. These loads will produce a reaction, $(\frac{7}{8}+\frac{6}{8}+\frac{5}{8})wp=2.25\ wp$, at the left support, and a reaction, $(3-2.25)wp=0.75\ wp$, at the right support. In the absence of counter dE, the corresponding live-load stress in De will be $S_l = (-)0.75\ wp \times h'/h$. The resultant stress in *De* is $S_d - S_l$. If $S_l < S_d$, the resultant stress is tensile, and the counter will not come into play. If $S_l > S_d$, the resultant stress is compressive, and if *De* is a flexible member, i.e., incapable of developing compression, this stress must be borne as a tensile stress by the counter, i.e.,

$$\text{Stress in counter } dE = S_l - S_d.$$

Initial Stress in Counters. Let it be assumed now that a tensile initial stress, S_i, has been induced in counter *dE*, by screwing up the turnbuckles. As shown in Art. 74, an equal tensile initial stress will thus be induced in the main diagonal *De*. Under dead loading the stresses in the diagonals cut by section 2 are then as shown in Fig. (*b*).



(*a*) (*b*)

FIG. 2.

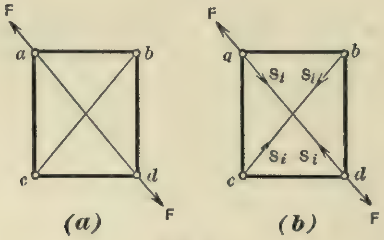
(*a*) *One Diagonal in Action.* Let the three live apex loads, *wp,* now be applied. The shear, 0.75 *wp,* from these loads will be resisted by a compressive stress in *De* and a tensile stress in *dE*. The sum of these two stresses will be $0.75\ wp \times h'/h = S_l$. Suppose the live-load compression in *De* is sufficient to neutralize the tension $S_d + S_i$. Since *De* is a flexible member it cannot resist any further compression. The live-load tensile stress in *dE* must then be $S_l - (S_d + S_i)$ which must be added to the initial stress S_i in that member. Thus

$$\text{Stress in counter } dE = S_l - (S_d + S_i) + S_i = S_l - S_d.$$

This value is seen to be the same as that previously found. Therefore under the conditions here assumed, the resultant stress in the counter is unaffected by its initial stress.

It may be shown similarly that the initial stress in the main diagonal does not increase the resultant stress in that member from dead and live loading, provided the initial tension in the counter is neutralized by the loading under consideration.

(*b*) *Both Diagonals in Action.* Unless the tensile stress in one of the two diagonals in a given panel is neutralized by the action of the live load, the stresses are statically indeterminate, as may be seen from the following simple illustration. Let it be assumed that the two rectangular frames in Fig. 3 are hinged at the corners, that the diagonals are flexible members, and that the initial stress in the diagonals is zero in Fig. (*a*) and S_i in Fig. (*b*). Let two equal and opposite forces be applied at *a* and *d*, in the directions shown, and let these forces be gradually increased from zero to F.



(*a*) (*b*)

FIG. 3.

In Fig. (*a*) the forces F tend to lengthen *ad* and shorten *cb*, i.e., to produce tension in the former and compression in the latter. Since *cb* is a flexible member without initial tension, it cannot develop compression. Hence stress *cb* as well as the stresses in the sides of the frame are zero, and stress *ad* equals F.

In Fig. (*b*) the sides of the frame are initially in compression. As the forces F are gradually applied they tend to produce tension in *ad* and compression in *cb*. The magnitudes of these stresses depend, however, on the elastic deformation of the frame, which is in turn dependent on the sectional areas of the members, assuming E, the modulus of elasticity, to be constant. Let it be assumed that F has such a value that the residual tension in *cb* is S_r. As shown in Art. 74, this stress must be balanced by an equal tensile stress, S_r, in the other diagonal, *ad*, and the stress in the latter is therefore $F+S_r$. As F is increased, S_r diminishes, and when S_r reaches zero the compressive stresses in the sides of the frame also become zero and stress *ad* equals F. For that and higher values of F the conditions in Fig. (*b*) are therefore precisely the same as in Fig. (*a*).

Reverting now to Fig. 2 (*b*), if the live-load compression in *De* is not sufficient to neutralize the tension S_d+S_i, let S_r denote the residual stress. The effect of this stress on the counter is the same as that of an initial stress S_r in *De*, which would induce an equal stress in the counter. Hence in this case

$$\text{Stress in counter } dE=(S_l-S_d)+S_r.$$

If $S_l=0$, $S_r=S_d+S_i$ and the stress in the counter by the above equation is S_i, as it should be. Again, the stress in the counter is greater or less than S_r according to whether $S_l>S_d$ or $<S_d$; whereas with no initial tension in the counter, it has been shown that for $S_l>S_d$, S_r becomes zero, and the stress in the counter is S_l-S_d; while for $S_l<S_d$ the stress in the counter is zero.

Thus it is seen that if the live-load compression is not sufficient to neutralize the tension, S_d+S_i, in the main diagonal, the stress in the counter is increased by initial stress. The amount of that increase depends on the value of S_r, which is statically indeterminate, and in fact the exact value of the initial stress S_i is itself unknown.

Similarly, if when the load is placed in position for maximum tension, S_l', in the main diagonal the initial tension in the counter is not wholly neutralized, the residual stress S_r' induces a like stress in the main diagonal. The resultant stress in the latter is then $S_l'+S_d+S_r'$, in which S_r' is again statically indeterminate. It is to be observed also that practically the joints, especially those in the upper chord, are not freely hinged, as assumed, which adds to the uncertainty of the stress distribution.

This analysis serves to show, however, (1) that moderate initial stresses induced by tightening the counters produce either no effect or only a slight effect on the resultant maximum stresses in the counters and main diagonals; (2) that when the initial stress becomes a factor in the determination of the maximum stress in a member, that stress is statically indeterminate; and (3) that excessive initial stresses should be avoided.

B. STRESSES FROM DEAD LOAD

82. General Method. The dead-load stresses in trusses with horizontal chords may in general be most conveniently found by the method of coefficients explained in Art. 59. By this method the stress in each member is expressed as the product of a constant and a numeric coefficient; or, in some cases, as will be seen presently, as the sum of two such products. This method is more rapid and convenient than the graphic method, especially with the aid of a slide rule.

83. Pratt Truss. A seven-panel through bridge of the Pratt type, Fig. 4, will be assumed. The numeric values of the upper and lower apex loads, W' and W_1, may be found in any given case as explained in Art. 79. The dead-load stresses in the counters, represented by broken lines, are zero (Art. 81). For present purposes the existence of these counters may therefore be disregarded.

(*a*) *Web Stresses. Diagonals.* For brevity, let $W = W' + W_1$; then $R = 3W$. As explained in Art. 59, the coefficients for the diagonals, Fig. 4, represent the shears in corresponding panels for $W = 1$. The shears for the actual loads are the products of these coefficients and W, and the stresses are the products of these shears and the secant of the angle θ between the diagonals and a vertical. This secant can be most conveniently expressed as the ratio h'/h, as in Art. 59. The stresses in the diagonals are thus the products of the coefficients shown in the figure and Wh'/h. The character of the stresses, distinguished by + and −, may be determined in the usual way by considering the external forces to either side of successive sections cutting three members.

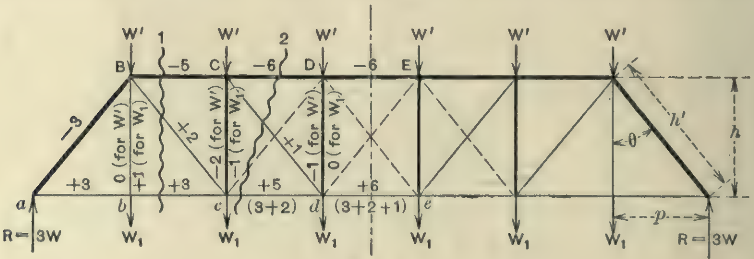


FIG. 4.

Verticals. The coefficients for the verticals may be found by analyzing the forces at suitable joints. Thus from joint b, the stress in the hip vertical Bb is seen to be $+W_1$. From the upper-chord joints, it is seen that the compressive stress in each vertical post is equal to the vertical component of the stress in the diagonal meeting the post in that chord, plus W'. The vertical component of the stress in any diagonal is the product of W and the coefficient of that diagonal. The stresses in the verticals are the sums of the products found by multiplying W' and W_1 by the appropriate coefficients shown in the figure.

(*b*) *Chord Stresses.* The coefficients for the chords are found as in Art. 59, and the stresses in these members are the products of these coefficients and Wp/h.

84. Practical Hints. The constants Wh'/h for the diagonal web stresses and Wp/h for the chord stresses may be evaluated by a single setting of the slide rule to the common factor W/h. The numeric coefficient should be written on each diagonal and lower-chord member. The stress in any member is then the product of its coefficient and the appropriate constant.

Since the loads are all 'counterbalanced' (Art. 80), the shear in each panel may be most conveniently found as the sum of the loads between that panel and the center of the truss, instead of as the algebraic sum of all forces to the left or right of the panel. If the truss has an even number of panels one-half the apex loads at the center should be included in such summations.

By expressing each chord stress as the sum of the preceding one and the horizontal components of the web stresses centering at the intervening joint, and by then applying an independent check to the greatest chord stress, the correctness of all the chord stresses may be verified. Thus

$$\left.\begin{aligned} ab &= bc = +3Wp/h \\ cd &= bc + 2 \quad \text{"} \\ de &= cd + 1 \quad \text{"} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The value of the greatest chord stress may then be checked as follows:

If the truss has an even number of panels, the greatest chord stress, which exists in this case in the central member of the upper chord, may be found by dividing the bending moment at the center of the truss by the height of the truss (Art. 57). If w equals the uniform dead load per foot of truss, the bending moment in question equals $\frac{1}{8}wl^2$, whence,

$$\text{Maximum chord stress} = \frac{wl^2}{8h}. \quad . \quad . \quad . \quad . \quad (2)$$

If the truss has an odd number of panels, as in the present case, the greatest chord stress may be found by dividing the bending moment at a section distant $\frac{1}{2}p$ from the center of the truss (Eq. (23), Art. 42) by the height of the truss, whence

$$\text{Maximum chord stress} = \frac{w(l^2 - p^2)}{8h}. \quad . \quad . \quad . \quad . \quad (3)$$

Applying this equation to the derivation of stress $de = CD = DE$, and noting that $l = 7p$ and $wp = W$,

$$de = +6W\frac{p}{h},$$

which agrees with the value found above.

Problem 1. The dead load is sometimes assumed to be applied wholly at the joints of the 'loaded' chord. Show (1) that the stresses in the verticals are alone affected by this assumption; and (2) that the errors are on the side of danger for through bridges, and on the side of safety for deck bridges.

Problem 2. Verify the coefficients in Fig. 5, for a truss of 10 equal panels.

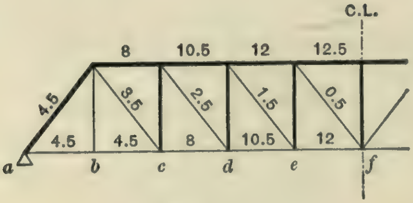


FIG. 5.

Problem 3. Given the following data for a through highway bridge, assume the entire dead weight to be concentrated at the lower chord joints and determine the stresses, (1) by the method of coefficients, and (2) by the graphic method. Compare results and time required by each method.

Span = 128 ft.; panel length = 16 ft.; height of truss = 20 ft.; weight of steel (trusses, floor beams and lateral bracing) = 300 lbs. per foot of bridge; weight of timber (joists, flooring and wheel-guards) and railing =450 lbs. per foot of bridge.

85. Chord Stresses by Moments. The chord stresses may also be conveniently found by moments. Taking sections 1, 2, etc., Fig. 4, Art. 83, the moment centers for stresses ab $(=bc)$, cd and de are located at B, C and D, respectively. Taking moments of all forces to the left of each section in turn, the stresses are:

$$\left.\begin{aligned}&ab=bc=(3W\times p)\div h;\\&cd=BC=(3W\times 2p-Wp)\div h;\\&de=CD=DE=(3W\times 3p-W\times\overline{p+2p})\div h\end{aligned}\right\}\quad . \quad . \quad . \quad (4)$$

Algebraically these expressions may be reduced to the same form as in Eq. (1), Art. 84, by the method of coefficients. The lower-chord stresses are tensile, the upper-chord stresses compressive. The equality of stresses cd and BC, for example, may be seen from the fact that their moment centers lie at C and c, respectively, i.e., in the same vertical line. Since the forces are all vertical, the moments of the forces to the left of section 2 about C and of section 1 about c, are equal.

General Formula for Chord Stresses. A general formula for chord stresses may be derived as follows:

Let n equal the number of panels from the end of the truss to the moment center of any chord stress. Then,

$$\text{Chord stress}=[R\times np-(n-1)W\times\tfrac{1}{2}np]\div h. \quad . \quad . \quad (5)$$

Designating the total number of panels in the truss by N, $R=\frac{1}{2}(N-1)W$, and Eq. (5) becomes:

$$\text{Chord stress}=n(N-n)\frac{Wp}{2h}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which n is the number of panels from the moment center of the chord stress to *either* end of the truss. If, for example, n is taken as the number of panels to the left support, $N-n$ will be the number of panels to the right support. Either of these values substituted for n in Eq. (6) will give the same result. Since the factor $Wp/2h$ in this equation is a constant for any given truss, the chord stresses may be found just as conveniently by this formula as by the method of coefficients.

Thus the chord stresses in the truss shown in Fig. 4, Art. 83, are:

$$\left.\begin{aligned} ab=bc&=+(1\times\overline{7-1}=\ \ 6)\,\frac{Wp}{2h}\\ cd&=+(2\times\overline{7-2}=10)\,\frac{Wp}{2h}\\ de&=+(3\times\overline{7-3}=12)\,\frac{Wp}{2h}\end{aligned}\right\}\quad\ldots\ldots\quad(7)$$

These expressions are similar in form and equal in value to those in Eq. (1), Art. 84, by the method of coefficients.

86. Howe Truss. The Howe truss, Fig. 6, differs from the Pratt truss in respect to the following features: (1) The main diagonals are inclined in the opposite direction, and are therefore compression members instead of tension members; (2) the verticals are in tension instead of in compression; and (3) the hip verticals, *Bb*, form an integral part of the main web system, whereas, in the Pratt truss, Fig. 4, Art. 83, these members are hangers carrying single lower apex loads.

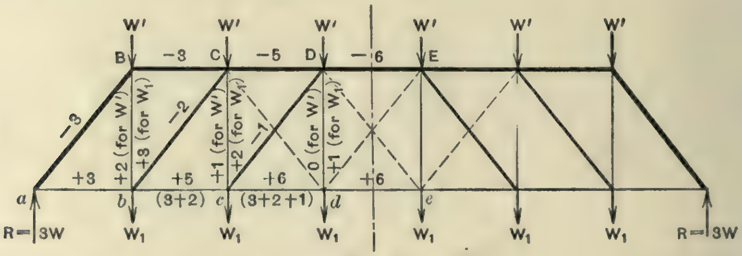


FIG. 6.

The Howe truss is used only for wooden or 'combination' bridges, the diagonals being made of wood and the verticals of steel. The end connections of the diagonals are so designed that these members are incapable of developing tensile stress. The counters, represented by broken lines, act in compression under certain conditions of partial live loading to be considered later, but the dead load produces no stress in these members.

The coefficients for the diagonals and chord members are found as explained for the Pratt truss.

The coefficients for the verticals may be found by considering the vertical components of the forces at the lower-chord joints.

87. Triangular Truss without Verticals. Referring to Fig. 7, the apex load at the hip joint B will be assumed equal to the intermediate upper apex loads. Strictly speaking, for the assumed condition of uniform distribution of the dead load, its value is $\frac{3}{4}W'$. For if w' denotes the dead load on the upper chord per foot of length, and $2p$ the panel length, the apex load at the hip joint will be $w'(\frac{1}{2}p+p)$ and the intermediate apex loads, $w'(p+p)$. The slight error involved is on the side of safety, and affects only the stresses in the inclined end posts and the chord members. If desired, for the stresses in these members, the coefficients involving the term W' may be corrected by subtracting 0.25 in each case. The assumed uniform distribution of dead load is not sufficiently accurate, however, to warrant such refinement.

(*a*) *Web Stresses.* The coefficients for W' are written above and those for W_1 below each member. Since the loads are counterbalanced, these values may be found as explained in Art. 84. Members sloping downward away from the center of the truss are in compression, and members sloping upward are in tension. The stresses are:

$$\left.\begin{aligned} aB &= -(2.5W'+2W_1)\frac{h'}{h} \\ Bb &= +(1.5W'+2W_1)\frac{h'}{h} \\ Cb &= -(1.5W'+W_1)\frac{h'}{h} \\ Cc &= +(0.5W'+W_1)\frac{h'}{h} \\ Dc &= -(0.5W')\frac{h'}{h} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

The web members in trusses of this type are usually given an equal inclination, as here assumed. For trusses having an odd number of panels that is a necessary condition for symmetry in the middle panel. In the general though uncommon case in which the web members have different inclinations, the panels being equal or unequal, the stress in any web member may be found as the product of the shear in the corresponding panel by the length of the member divided by the height of the truss.

(*b*) *Chord Stresses.* Since the web members meeting at any apex lie on opposite sides of a vertical line through the apex, and their stresses are opposite in character, their horizontal components are exerted in like direction, and always away from the center of the truss for lower-chord joints, and toward the center for upper-chord joints, as may be seen from the directions of the arrow-heads in Fig. 7. The chord stresses are therefore cumulative from the ends to the center of the truss. The coefficients for W' are written above and those for W_1 below each member. The stresses are:

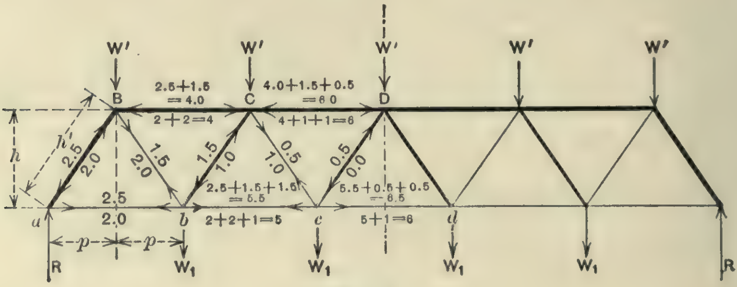


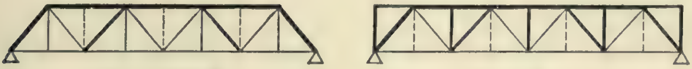
FIG. 7.

$$
\left.\begin{aligned}
ab &= +(2.5W' + 2W_1)\frac{p}{h} \\
BC &= -(4W' + 4W_1)\frac{p}{h} \\
bc &= +(5.5W' + 5W_1)\frac{p}{h} \\
CD &= -(6W' + 6W_1)\frac{p}{h} \\
cd &= +(6.5W' + 6W_1)\frac{p}{h}
\end{aligned}\right\} \quad \dots\dots \quad (9)
$$

Problem. Find the chord stresses in the triangular truss shown in Fig. 7, by the method of moments, Art. 85.

88. Triangular Truss with Verticals. The triangular truss is usually designed with verticals carrying intermediate floor beams. These verticals are represented by full lines in Fig. 8.

Verticals supporting the unloaded chord at intermediate points, shown by dotted lines, are also generally used.



(a) Through (b) Deck

FIG. 8.

(a) *Web Stresses.* Fig. 9 represents one-half of a truss of ten equal panels. The coefficients for W' and W_1 are equal, as for the Pratt truss, Art. 83, and the Howe truss, Art. 86. Their values, obtained as previously explained, are shown in the figure.

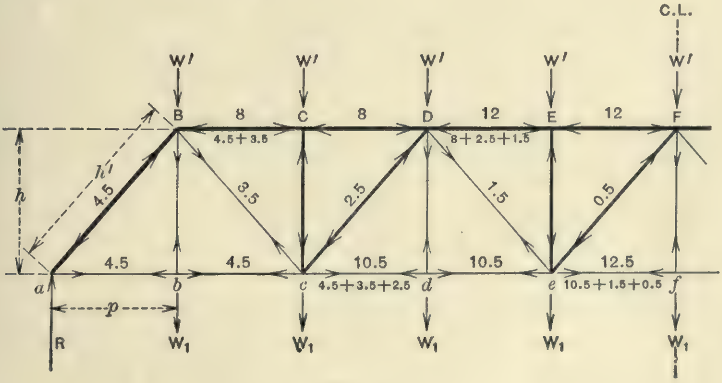


FIG. 9.

The diagonals sloping downward away from the center of the truss are in compression, and those sloping upward, in tension. Letting $W=W'+W_1$, the stresses in the diagonals are:

$$\left.\begin{aligned} aB &= -4.5Wh'/h \\ Bc &= +3.5 \quad \text{''} \\ cD &= -2.5 \quad \text{''} \\ De &= +1.5 \quad \text{''} \\ eF &= -0.5 \quad \text{''} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (10)$$

The stresses in the verticals are obviously equal to single apex loads. The stress in the hangers Bb, Dd and Ff is $+W_1$, and that in the struts Cc and Ee is $-W'$.

(*b*) *Chord Stresses.* The coefficients for the chord stresses, found as explained in Art. 59, are shown in the figure. These stresses are:

$$\left.\begin{aligned} ab = bc &= +\ 4.5Wp/h \\ BC = CD &= \quad -8 \quad\ '' \\ cd = de &= +10.5 \quad\ '' \\ DE = EF &= \quad -12 \quad\ '' \\ ef &= +12.5 \quad\ '' \end{aligned}\right\} \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad (11)$$

Problem. Find the chord stresses in the triangular truss shown in Fig. 9, by the method of moments, Art. 85.

89. Whipple Truss. The Whipple truss is now seldom built, although until about 1890 it was the prevailing type for trusses exceeding 175 to 200 ft. in length. The analysis of this truss will nevertheless be considered in some detail, because it involves certain interesting and instructive features, applicable to redundant truss systems in general, and also because the stresses in existing Whipple trusses have frequently to be considered with a view of determining their safety under current conditions of loading.

The Whipple truss, Fig. 10 (*a*), may be regarded as a ‘composite’ truss consisting of two Pratt trusses with separate web systems (except in respect to the inclined end posts and the hip verticals *Bb*) and with common chords. The stresses are found by assuming the two component trusses to act independently, each carrying only the loads at its own joints. The stresses in members forming part of both trusses, i.e., the chords and end posts, are then found by addition.

90. Truss with Even Number of Panels. One-half of a truss of 12 equal panels is shown in Fig. 10 (*a*). Since the loads are all counterbalanced they produce no stress in the counters, represented by dotted lines, and these members may be disregarded. The truss may then be divided into the two component trusses shown in Figs. (*b*) and (*c*). The hip verticals are assumed to be part of the latter, which will also be assumed to carry the loads W' at the hip joints. The hip verticals may be considered as belonging to either truss, without affecting the resultant stresses in the composite truss. Again, these members may be included with both trusses and the loads at joints *B* and *b* may then be divided equally, or in any other proportion, between the two trusses. The resultant stresses will be unaffected provided the loads assumed on each component truss are counterbalanced. If, however, one hip vertical is included with each truss, or if both are included with each truss and the loads on each truss are assumed unbalanced, the resultant stresses are incorrect, and also inconsistent with the assumed condition of no stress in the counters, as will be shown in the next article.

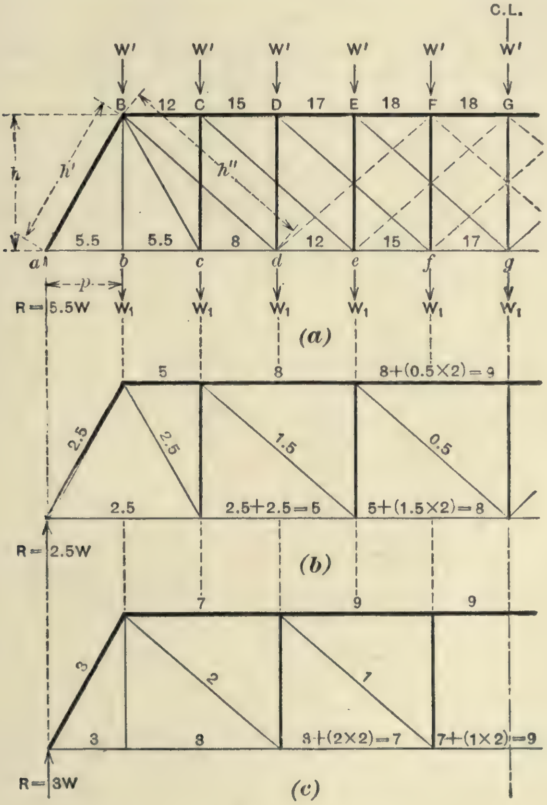


FIG. 10.

Disregarding counters, the stresses in a Whipple truss from counterbalanced loading are statically determinate, provided the number of panels is even, there being then no redundant members. In the criterion for redundancy, Art. 47, $m=2(j-3)+3$, in which, from Fig. (*a*), exclusive of counters, $m=45$ and $j=24$, so that the criterion is satisfied.

(*a*) *Web Stresses.* Since the loads on each web system, Figs. (*b*) and (*c*), are counterbalanced, the coefficients for the diagonals may be found as for the Pratt truss, Art. 83. Letting $W=W'+W_1$, and noting that since the inclined end post, *aB*, forms part of each system, the coefficients for that member must be added, the stresses in the diagonals are as follows:

$$\left.\begin{aligned} aB &= -5.5Wh'/h \\ Bc &= +2.5 \quad \text{``} \\ Bd &= +\ \ 2Wh''/h \\ Ce &= +1.5 \quad \text{``} \\ Df &= +\ \ 1 \quad \text{``} \\ Eg &= +0.5 \quad \text{``} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The stresses in the verticals are found as explained for the Pratt truss, Art. 83. These stresses are:

$$\left.\begin{aligned} Bb &= +W_1 \\ Cc &= -(1.5W+W') \\ Dd &= -(W+W') \\ Ee &= -(0.5W+W') \\ Ff &= Gg = -W' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(*b*) *Chord Stresses.* The coefficients for the chord stresses, shown in Fig. (*a*), represent the sums of the coefficients for the corresponding members in Figs. (*b*) and (*c*). The latter are found in the same way as in previous cases, noting only that if the stresses in two diagonals of length h' and h'' respectively have the same vertical components, the horizontal component of the latter is twice as great as that of the former. The coefficients in Figs. (*b*) and (*c*) corresponding to the horizontal components of diagonals crossing two panels are therefore doubled, as indicated in these figures. The chord stresses are obtained by multiplying the coefficient in Fig. (*a*) by Wp/h, the lower chord being in tension and the upper in compression. After the coefficients for web stresses have been obtained the coefficients for the chord stresses may be found more directly by applying $\Sigma H = 0$ to each joint of the composite truss in turn, beginning at the hip joint.

91. Methods of Analysis Compared. It was stated in the preceding article that if one hip vertical is included in each component truss the resulting stresses are incorrect and inconsistent with the assumed condition of no stress in the counters. This may be illustrated as follows: If the truss shown in Fig. 11 (*a*) is loaded with two counterbalanced weights, W, it is stable without the diagonal counters. The coefficients for the stresses are indicated in Fig. (*b*). If the truss is separated into two component trusses with web systems as distinguished by full and broken lines in Fig. (*c*), each system carrying a single load, the coefficients for each system will have the values shown in that figure. The resultant coefficients for the composite truss are given in Fig. (*d*). The analysis is thus seen to lead to the conclusion that the counters are subject to stress, and the chord stresses in the upper and lower middle panels are affected accordingly, as may be seen by comparing the coefficients in Fig. (*d*) with those in Fig. (*b*). The stresses indicated by the coefficients in Fig. (*d*) meet the requirements for equilibrium, but they are nevertheless incorrect and inconsistent with the generally assumed condition governing counters, namely, that they are so adjusted as not to come into play

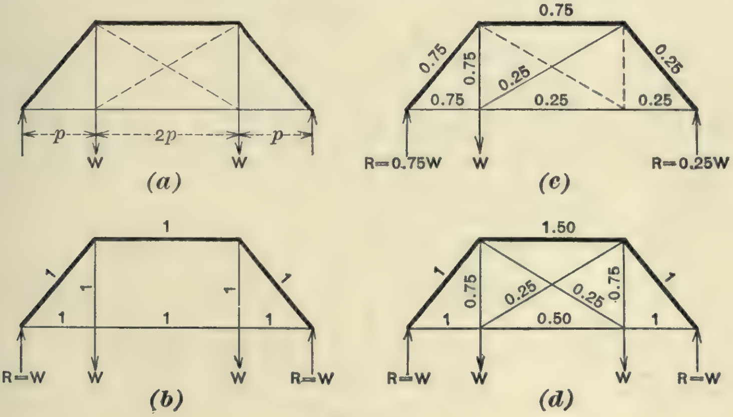


Fig. 11.

under counterbalanced loading. While the maintenance of such adjustment cannot be depended on in practice, and counters are, in fact, usually adjusted to some initial tension, the stresses in trusses containing counters are statically indeterminate under any other assumption (Art. 81*b*). That is to say, if the counters do act under the counterbalanced loading assumed in this case, the stresses cannot be determined by the method used above, but will depend then on the elastic deformation of the truss, as will be shown in the next article.

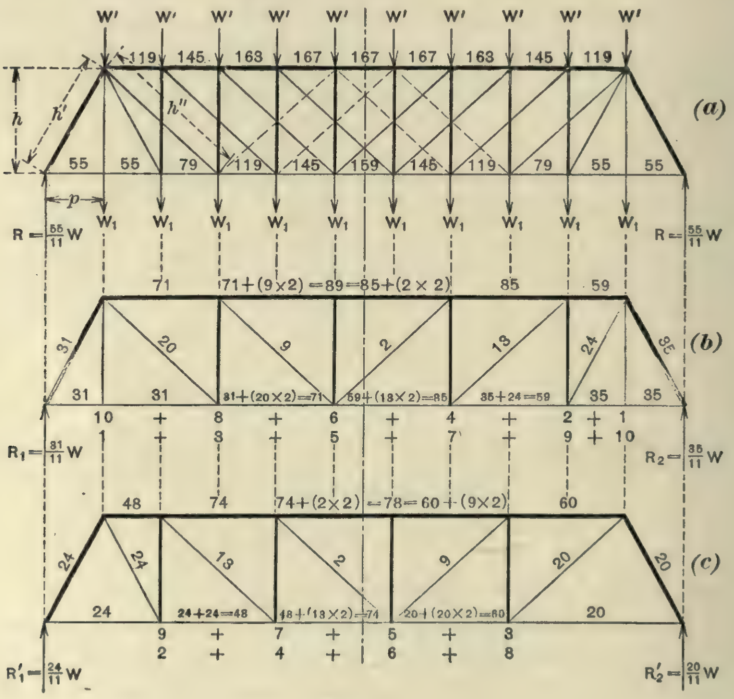


FIG. 12.

Reverting to the truss shown in Fig. 10 (*a*), if the left hip vertical and the load W' at B were included in the upper component truss, Fig. (*b*), and the right hip vertical and corresponding load W' in the lower truss, Fig. (*c*), the analysis would indicate a stress in the counter sloping from F downward to the right, with resulting slight changes in all stresses except in members aB, Bb, ab and bc. If the right hip vertical and load W' were included in the upper and the left in the lower truss, the counter sloping from f upward to the right would be brought into action.

92. Truss with Odd Number of Panels. (*a*) *First Approximate Method.* If the number of panels in a Whipple truss is odd, the stresses are statically indeterminate, even for counterbalanced loading, and the stresses obtained by the method employed in Art. 90, for a truss with an even number of panels, are only approximately correct.

Assume a truss with 11 equal panels, Fig. 12 (*a*), and let it be separated into the two component trusses, Figs. (*b*) and (*c*), including both hip verticals in either of these trusses, but in the same truss, for reasons explained in the preceding article. Since the loading is counterbalanced, the counters are omitted. The web systems are seen to be unsymmetrical. Letting $W=W'+W_1$, the reaction R_1, Fig. (*b*), equals $(10+8+6+4+2+1)W/11$. The figures in the parentheses are shown at successive joints. The lower figures represent similar terms for R_2. By thus computing R_1 and R_2 separately, and noting that their sum should be $6W$, the numerical work is checked. The reactions R_1' and R_2', Fig. (*c*), are found similarly. Again, the reactions R, Fig. (*a*), must equal $5W$, and must be equal to the sums of the corresponding reactions in Figs. (*b*) and (*c*).

Thus, $$R=R_1+R_1'=\frac{31}{11}W+\frac{24}{11}W=5W,$$

and $$R=R_2+R_2'=\frac{35}{11}W+\frac{20}{11}W=5W.$$

Web Stresses. The coefficients written on the diagonals, Figs. (*b*) and (*c*), represent the shears in corresponding panels in terms of $\frac{1}{11}W$. They are found, therefore, by beginning at each end, using the numerator of the reaction at that end, and subtracting 11 for each successive load. The stresses in the inclined end posts are $5Wh'/h$. The stresses in the other diagonals are found as the products of the coefficients and $\frac{1}{11}Wh'/h$ or $\frac{1}{11}Wh''/h$, according to whether the diagonal crosses one or two panels.

The stresses in the verticals (not given) are found as in Art. 90. Since the truss as a whole is symmetrical and the loading is counterbalanced, the stresses in verticals symmetrical about the center are equal.

Chord Stresses. The coefficients for the chord stresses are found as explained in Art. 90, proceeding from each end of the component trusses to the apex at which the slope of the diagonals changes, and checking the values on opposite sides of this point. Thus, in Fig. (*b*), $71+2(9-2)=85$, and in Fig. (*c*), $74+2(2-9)=60$. The resultant coefficients, shown in Fig. (*a*), are the sums of those in Figs. (*b*) and (*c*). The stresses are obtained by multiplying these coefficients by $\frac{1}{11}Wp/h$. The stresses in chord members symmetrical about the center are equal, for the same reason as for the verticals. This serves as a final check on the correctness of the values.

(*b*) *Second Approximate Method.* That the Whipple truss with an odd number of panels is statically indeterminate, even for counterbalanced loading, for which the counters are disregarded, is due to the presence of the main diagonals intersecting the middle panel. Since the shear in that panel for counterbalanced loading is zero, these members are not needed for stability. If these members are treated as counters and if it is assumed that they are so adjusted as to render them incapable of developing stress under counterbalanced loads, the stresses under that form of loading become statically determinate. The truss, Fig. 13 (*a*), may in that case be separated into the two symmetrical component systems shown in Figs. (*b*) and (*c*). The reactions, R_1 and R_1', are then equal for each truss, and the coefficients for the various stresses, shown in these figures, are found in the usual manner. The resultant coefficients for the composite truss are shown in parentheses in Fig. (*a*), in which the values of the coefficients found by the previous method are also shown above each member for convenient comparison. It is seen that the differences are not great, and they will become relatively less with an increasing number of panels.

The Whipple truss is rarely used at present, but during the period when it was the prevailing type for spans exceeding 175 ft. in length, the dead-load stresses for trusses containing an odd number of panels were commonly computed by method (*a*), and that method may be regarded as preferable to method (*b*).

(*c*) *Exact Method.* The error involved in the neglect of the diagonals crossing the middle panel, unless they are so adjusted as to hang slack under counterbalanced loading, may be readily shown by the following example. In the five-panel Whipple truss, Fig. 14, assume the sectional areas of all members to be known. Assume the apex loads W to produce unit stresses of 12,000 lbs. per square inch in the compression members and 15,000 lbs. per square inch in the tension members, if the diagonals Bd and Ec are disregarded in computing the stresses. Then if the modulus of elasticity E equals 30,000,000, the members forming the sides of the trapezoid $BcdE$ will suffer the deformations in inches shown in Fig. 14, the stresses Cc and Dd being zero. These deformations cause elongations of 0.115 in. in the diagonals

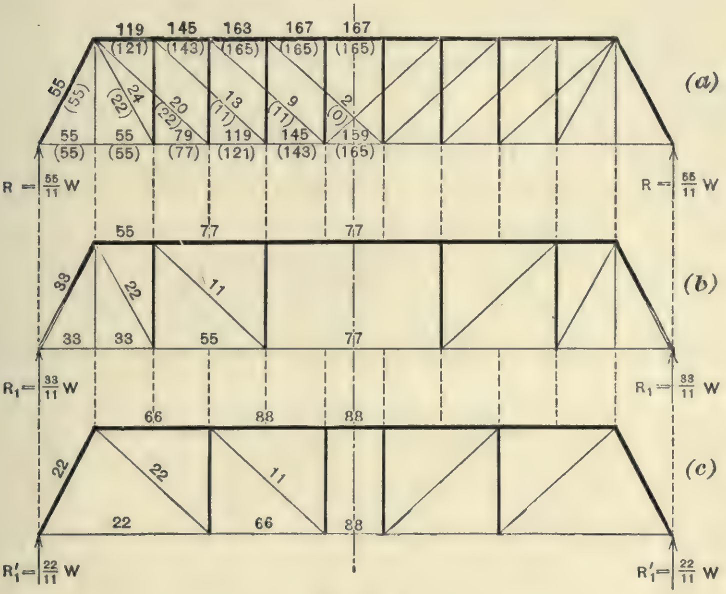


FIG. 13.

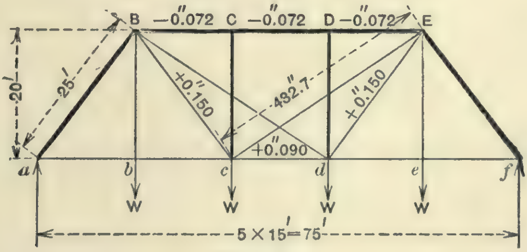


FIG. 14.

Bd and *Ec* of the trapezoid, which will develop unit stresses of $30{,}000{,}000 \times 0.115/432.7 = 8000$ lbs. per square inch, if these members are so adjusted as to come into play immediately on the application of any part of the assumed loading, and if their sectional areas, a, are relatively so small that the total stresses, $8000 \times a$, will not seriously influence the stresses, and hence the deformations, of the other members. Otherwise the latter will have to be recomputed for an assumed stress of $8000\,a$ in the diagonals, and the process repeated. The value of the stresses *Bd* and *Ec* will be somewhat reduced by this second approximation.

Again, Fig. 15 represents a seven-panel Whipple truss in which all stresses were first computed for apex loads of unity along the lower chord. The stresses were first found by the approximate method (*a*) and the sectional areas proportioned for unit stresses of 12,000 and 15,000 lbs. per square inch in compression and tension members respectively. The *exact* stresses were then found by a method involving the consideration of the elastic deformations of all members under stress. The figures above each member denote the stresses by method (*a*) and the values in parentheses the stresses by method (*b*), both in percentages of the exact stresses. The differences between the two sets of figures are seen to be comparatively small, except for the middle diagonals, and they exhibit the same general relations as those in Fig. 13. In the latter the differences are relatively less since the number of panels is greater. It is also seen from the values in Fig. 15 that the exact stresses are *intermediate* between those obtained by the two approximate methods.

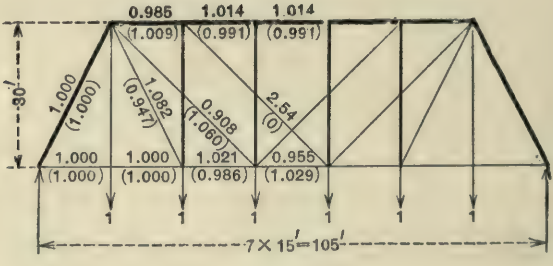


FIG. 15.

93. Double Triangular Truss. The stresses in this truss, Fig. 16 (*a*), are statically determinate only if the number of panels is even and the loads are counterbalanced. Assuming that these conditions are fulfilled in the case of the truss of which one-half is shown in the figure, it follows, since the truss as well as the loading is symmetrical about the center line, that the stresses in symmetrically disposed members are equal. The vertical components of the stresses in the diagonals meeting at the center of the top chord are therefore $0.5W'$ and in those meeting at the center of the bottom chord, $0.5W_1$. The coefficients for all the remaining web members may then be found by analyzing the conditions at successive joints from the center. In case the loads are not counterbalanced, the stresses in none of the web members, except the inclined end posts, are statically determinate. Applying the criterion for redundancy, Art. 47, it is seen that since the truss has 20 joints, the number of members should be 37, whereas the actual number is 38. A little consideration will show that any one of the diagonals between

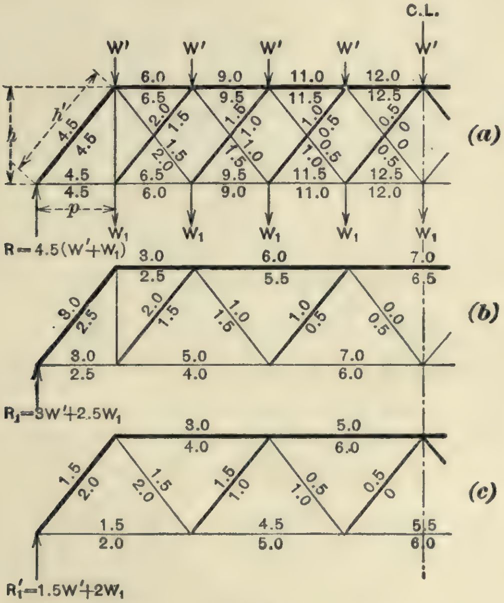


FIG. 16.

the hip verticals might be omitted without sacrificing the stability of the truss.

94. Truss with Even Number of Panels. The truss shown in Fig. 16 (*a*) may be separated into the two symmetrical component trusses (*b*) and (*c*). The loads W' at the hip joints will be included wholly with truss (*b*). These loads may be divided between the two trusses in any proportion without affecting the resultant stresses in the composite truss, provided the component loads assumed on each truss are counterbalanced. The coefficients for the various members, Figs. (*b*) and (*c*), are found in the usual way. The figures shown above the members are the coefficients for the upper loads, W', and the figures below the members are those for the lower loads, W_1. The coefficients for the component trusses combined give the values shown in Fig. (*a*). The stresses are then found in the usual manner. For the web stresses the upper coefficient for each member is multiplied by $W'h'/h$ and the lower coefficient by W_1h'/h. The stresses are then the sums of these products. For the chord stresses the corresponding factors are $W'p/h$ and W_1p/h. The diagonals sloping downward away from the center line of the truss are in compression, and those sloping in the opposite direction are in tension. The stress in the hip vertical is the sum of the vertical component of the stress in the diagonal meeting this member in the lower chord and the apex load W_1, i.e., $+(2W'+2.5W_1)$.

95. Truss with Odd Number of Panels. If the number of panels is odd, Fig. 17 (*a*), the stresses are statically indeterminate even for counterbalanced loads. Any one of the diagonal web members might be omitted without affecting the stability of the truss. If that is actually done, the truss will satisfy the criterion for redundancy, and the stresses will then be statically determinate under any distribution of loading. Under the conditions represented in Fig. (*a*) it is not possible to determine the stresses in any of the diagonals by independent considerations, as in the case of a truss with an even number of panels. A section severing the truss through any intermediate panel will traverse four members whose stresses are unknown, whereas only three conditions of equilibrium are available for their determination. Recourse must be had to approximate conventional methods for the analysis of the stresses. After the truss members have been proportioned in accordance with these stresses, the true stresses may be found by applying the exact method based on the elastic deformation of the members.

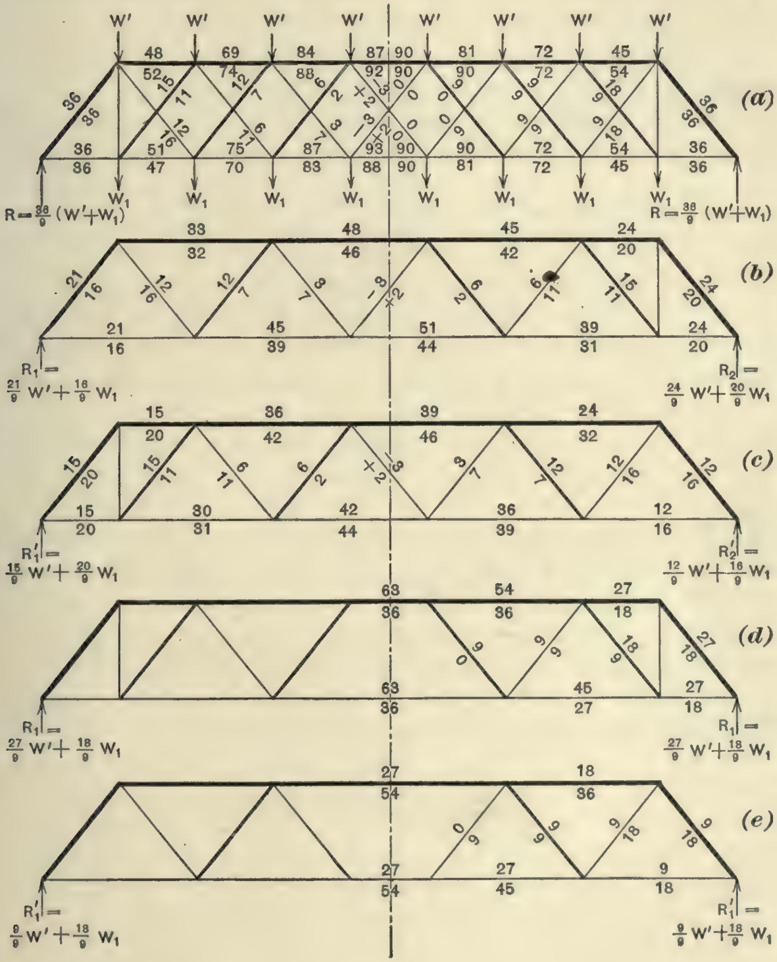


FIG. 17.

(*a*) *First Approximate Method.* The truss may be separated into the two component trusses shown in Figs. (*b*) and (*c*), the loads W' at the hip joint being included with either truss. Assuming these loads applied to the truss in Fig. (*b*), the coefficients shown are found in the usual way, the values above and below each member relating to W' and W_1, respectively. The resultant coefficients are those shown in Fig. (*a*) to the left of the center line. The stresses are obtained as explained in Art. 94, except that the factor $\frac{1}{8}$ is to be included with each coefficient in the present case.

(*b*) *Second Approximate Method.* Since the loads are counterbalanced, the shear in the middle panel, Fig. (*a*), is zero, so that for such loading the truss is stable without the middle diagonals. Assuming that these members are omitted, the truss may be separated into the two symmetrical component trusses shown in Figs. (*d*) and (*e*). Including the loads, W', at the hip joints with the upper truss, the coefficients have the values indicated. The resultant coefficients appear in Fig. (*a*) to the right of the center line. The application of this method is even less warranted in this case than for the Whipple truss, Art. 92, since the middle diagonals in the double triangular truss are counterbraced, and therefore cannot possibly hang slack under counterbalanced loading; whereas with flexible and adjustable counters that condition, however improbable, is at least a possible one.

Comparing the coefficients to the right and left of the center line in Fig. (*a*), it is seen, first, that the differences are relatively less for the chord stresses than for the web stresses; and second, that the differences in the coefficients for W' and W_1 respectively, are opposite in sign for each member.

(*c*) *Exact Method.* Fig. 18 represents a five-panel, double-triangular truss in which the exact stresses for apex loads of unity along the lower chord were found after the sectional areas of the members had been determined according to the stresses found by the approximate method (*a*), for unit stresses of

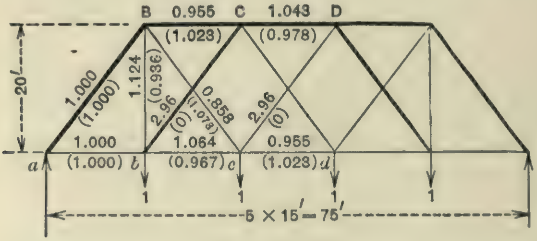


Fig. 18.

12,000 and 15,000 lbs. per square inch, in compression and tension, respectively. The figures above each member represent the stresses by method (*a*) and the values in parentheses the stresses by method (*b*), in percentages of the exact stresses. By method (*b*) the stresses in diagonals *Cb* and *Dc* are zero. The errors for these diagonals by method (*a*), although on the side of safety, are seen to be large. For the other web stresses the errors on the side of danger, by the approximate methods, range from about 6 to 14 per cent, and for the chord stresses from about 2 to 4.5 per cent.

96. Quadruple Triangular Truss. This truss, Fig. 19 (*a*), contains four component web systems Figs. (*b*), (*c*), (*d*) and (*e*). If the number of panels is even, as in the present case, in which there are 12 panels, two of the component trusses, Figs. (*b*) and (*d*), are symmetrical about the center line, and the other two are identical if turned end for end. If the number of panels is odd, none of the component trusses are symmetrical.

Since the composite truss, Fig. (*a*), contains 24 joints, the number of members should be 45 in order that the stresses may be statically determinate (Art. 47); whereas their actual number is 48, thus indicating the presence of 3 redundant members. The conditions are further complicated by the fact that the web members are riveted together at points of intersection and to the chords at their extremities, which gives rise to 'secondary' stresses from bending as the various members lengthen and shorten under stress. Since the web members are usually long and slender, these secondary stresses may be neglected without serious error.

(*a*) *First Approximate Method.* The loads at the hip joints may most conveniently be assumed as applied to one of the symmetrical trusses, Fig. (*b*) or (*d*). The former was chosen, although the resultant stresses will be the same under either assumption. The coefficients in the component truss are found in the usual way. The factor $\frac{1}{12}$ is to be understood as included with each value. As before, the numbers above and below each member relate to W' and W_1, respectively.

In Fig. (*b*), three members meet at the end supports, so that the coefficients cannot be found by beginning at that apex. On reference to the hip joint it is seen, however, that the vertical component of the stress in the inclined end post is W'. The coefficients for W' and W_1 for that member are therefore 12 and 0, respectively. In trusses (c) and (e) the upper and lower apex loads produce stresses opposite in character in the middle diagonal, that from the upper loads being compressive, and that from the lower, tensile. In obtaining the coefficients for the chord stresses regard must be had to the slope of the web members (Art. 90b) and to the character of the stresses in the latter.

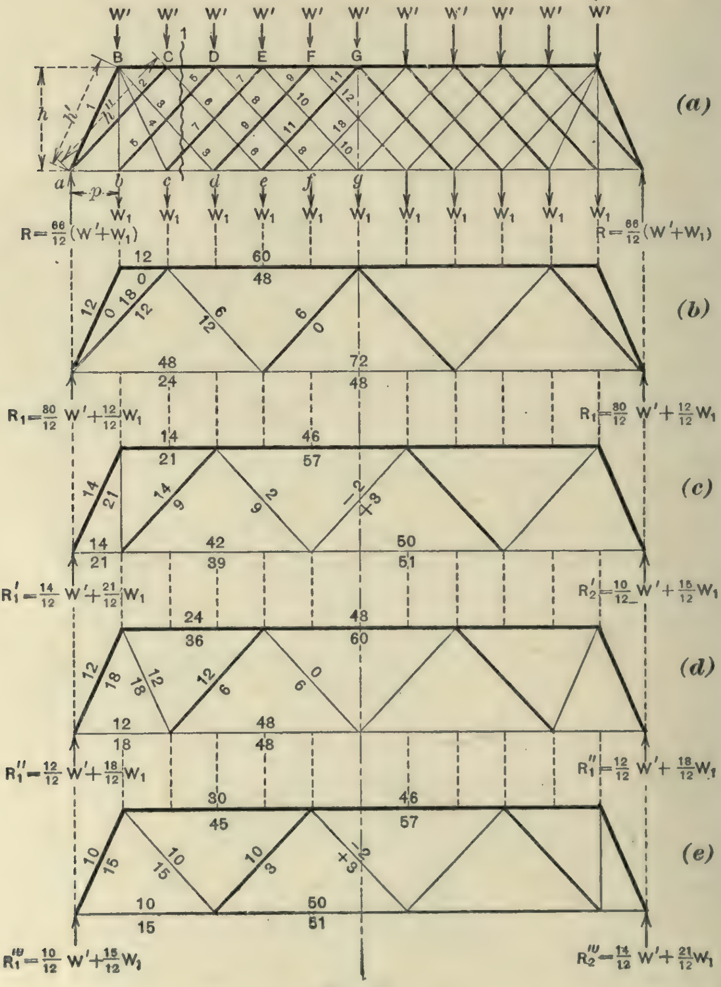


FIG. 19.

The stresses in web members sloping downward away from the center of the truss are compressive, and in those sloping upward away from the center, tensile. The difference between adjacent chord stresses is equal to the arithmetical sum or difference of the horizontal components of the stresses in the two web members meeting at the intervening apex, according to whether these stresses are opposite or alike in character. The resultant coefficients for the chord stresses are obtained by summing the coefficients for the component trusses. These coefficients as well as those for the web stresses are given in Table I.

TABLE I

COEFFICIENTS FOR QUADRUPLE TRIANGULAR TRUSS BY TWO APPROXIMATE METHODS

Web Members.	Coefficients.			Chord Members.	Coefficients.		
	First Method.		Second Method.		First Method.		Second Method.
	W'	W_1	W' and W_1		W'	W_1	W' and W_1
1	48	54	49.5	*ab*	84	78	82.5
2	18	12	16.5	*bc*	112	96	120
3	10	15	13.5	*cd*	148	126	162
4	12	18	13.5	*de*	188	162	192
5	14	9	13.5	*ef*	212	186	210
6	6	12	10.5	*fg*	220	198	216
7	12	6	10.5				
8	2	9	7.5	*BC*	80	102	120
9	10	3	7.5	*CD*	128	150	162
10	0	6	4.5	*DE*	160	186	192
11	6	0	4.5	*EF*	184	210	210
12	−2	+3	±1.5	*FG*	200	222	216
13	−2	+3	±1.5				

The resultant web stresses are found by multiplying the coefficients for W' and W_1 by $\frac{1}{12}W' \times (h'/h$ or $h''/h)$ and $\frac{1}{12}W_1 \times (h'/h$ or $h''/h)$, respectively, and adding these products for each member. The factors to be used for the chord-stress coefficients for W' and W_1 are $\frac{1}{12}W'p/h$ and $\frac{1}{12}W_1 p/h$, respectively.

(*b*) *Second Approximate Method.* The stresses are sometimes analyzed by considering (1) the shear in each panel to be equally divided among the web members traversing that panel, and (2) the bending moments to be resisted wholly by the chord stresses. These assumptions are so loose, however, that this method is not to be recommended.

If this method is applied to the analysis of the stresses in the truss shown in Fig. 19 (*a*), it is clear that the shear in the first panel, $\frac{66}{12}(W'+W_1)$, cannot properly be regarded as equally divided between web members 1 and 2. Since three of the four component web systems are supported at the hip joint, the shear in this panel should be approximately divided in the ratio of 3 to 1 between members 1 and 2, the resulting coefficients being 49.5 and 16.5.

Considering the shear, $\frac{54}{12}(W'+W_1)$, in the second panel to be equally divided between the web members 2, 3, 4 and 5 traversing that panel, the coefficient for each member will be 13.5. Obviously, for member 2 the higher value 16.5, previously found, will have to be retained. Similarly, the shear, $\frac{42}{12}(W'+W_1)$, in the third panel leads to coefficients 10.5 for members 3, 5, 6 and 7, but for members 3 and 5 the previously found value, 13.5, will govern. The resulting coefficients are shown in Table I, and they are seen to differ greatly from those found by the first method.

For chord stresses, the bending moments at successive panel divisions from the end, in terms of $(W'+W_1)p$, are 5.5, 10, 13.5, 16, 17.5 and 18. The coefficients for chord stresses in Table I were found by multiplying these values by 12, except that the coefficient for stress *ab* was obtained consistently with those for stresses 1 and 2 as $49.5+(16.5\times 2)=82.5$. The value $5.5\times 12 = 66$ for this member would have involved a large error on the side of danger. The differences in the coefficients for chord stresses by this method and the previous one are seen to be slightly on the side of danger for a few stresses, but usually greatly on the side of safety.

The assumption underlying this method, namely that the bending moments are resisted wholly by the chord stresses, involves gross errors from the neglect of the moments of the stresses in the web members severed in taking successive imaginary sections. Thus, taking section 1, Fig. 19 (*a*), and assuming the moment center for stress *cd* at *D*, the moments of web stresses 3, 6 and 7 about *D*, Fig. 20, are neglected, that is to say, the resultant moment of these stresses is assumed to be zero.

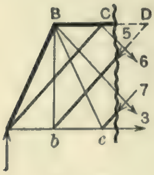


FIG. 20.

If the moment centers are assumed at mid-panel points instead of panel points, the resulting chord stresses will be in better average agreement with those found by the first method, but the differences on the side of danger will be much emphasized.

(c) *Exact Method.* After the truss has been designed in accordance with the stresses found by approximate method (*a*), the exact method of analysis, based on the elastic deformation of the members, may be applied. Since, however, there are in the present case three redundant members, the calculations are somewhat tedious. Moreover, since the web members are riveted together at points of intersection, and for other reasons which will not now be considered, it is doubtful whether the stresses thus computed will accord more closely with the actual stresses than those derived by method (*a*).

C. STRESSES FROM UNIFORM LIVE LOAD. CONVENTIONAL METHOD

97. General Considerations. The live-load stresses in the trusses of highway bridges and very long railway bridges are usually computed on the assumption that the live load is uniformly distributed. In deriving the live-load stress in any member the load is assumed to cover only such parts of the bridge as will cause the stress in that member to become maximum. In the case of very long railway bridges, the stresses in 'secondary' members which become maximum when the load extends over the two adjacent panels only, are usually based, however, on a system of loading consisting of a series of wheel loads representing two locomotives coupled, followed by a uniform train load. The maximum floor-beam reactions from such loading are found as explained in Art. 45.

98. Conventional Loading for Web Stresses. In trusses with horizontal chords, the web stresses are direct functions of the vertical shears. Considering the general case represented in Fig. 21, a load applied anywhere along the distance a will produce a positive shear equal to R in panel 5–7, i.e., compression in member 5–6 and tension in member 6–7. On the other hand, a load anywhere along the distance b will cause a negative shear equal to R_1 in panel 5–7, i.e., tension in member 5–6 and compression in member 6–7. A load W in panel 5–7 will produce a positive or negative shear, V, in that panel according to its position. Thus, if r equals the floor-beam reaction at 5(=load at apex 5) and R the reaction at the left end of the truss from W,

$$V=R-r$$

or

$$V=W\left(\frac{a+x}{l}-\frac{x}{p}\right).$$

Hence the shear is positive when $(a+x)/l>x/p$ and vice versa. If the load is so placed that $(a+x)/l=x/p$, the shear it produces in panel 5–7 is zero. That condition is met when

$$x=\frac{a}{l-p}\,p. \quad \ldots\ldots\ldots ✓ \ . \tag{14}$$

Eq. (14) is perfectly general and it shows that for maximum positive shear in any panel, the uniform live load must extend from the right support to a point in the panel distant $ap/(l-p)$ from its right end, and that for maximum negative shear in that panel the load must extend from the left support to the same point.

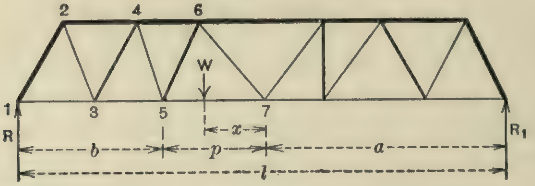


FIG. 21.

For the right end panel a=zero, which shows that a load anywhere in that panel produces negative shear in that panel. Eq. (14) also shows that for the usual case of equal panels (p constant) x increases directly with a. For the left end panel, for which $a=l-p$, $x=p$, indicating that a load anywhere in the left end panel produces positive shear in that panel, and that for maximum positive shear in the left end panel the load must extend over the entire span, as is also evident from independent considerations.

For the usual case of equal panels, a, l and p in the fraction may be conveniently expressed in terms of panel lengths. Thus if n and N denote the number of panels in a and l respectively,

$$x=\frac{n}{N-1}\,p. \quad \ldots\ldots\ldots ✓ \ . \ . \tag{15}$$

Assuming a truss of six equal panels, Fig. 22, the positive shear in panel 5–7, for example, becomes maximum when the load extends from the right support to a point q in the panel at a distance x from apex 7. From Eq. (15) $x=0.6\ p$. If the load is extended to apex 5, as shown in the figure, the positive shear in panel 5–7 will be reduced, since the load from q to 5 produces negative shear in that panel.

By the so-called conventional method for uniform loading the maximum stress in any given web member is found by assuming full joint loads wp to be applied at those joints which will cause the stress to become maximum in the member considered. If the truss is partially loaded, as in Fig. 22, a full panel load cannot exist at joint 7, for example, unless panel 5–7 is fully loaded, and in that case a forward apex load $\frac{1}{2}wp$ exists at 5. The method is conventional in that the existence of this forward load is ignored. On that assumption the shear in panel 5–7 equals R, which, in this case, equals $(\frac{1}{6}+\frac{2}{6}+\frac{3}{6})wp=wp$. The errors resulting by the use of this method are always on the side of safety, as will be shown hereafter (Art. 124).

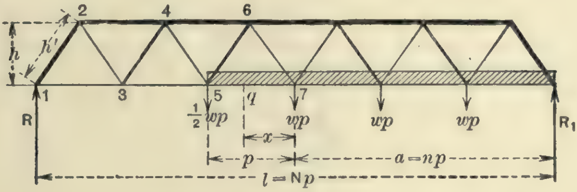


FIG. 22.

A general expression for the positive shear, V, in any panel by the conventional method may be derived by considering that it is equal to the left reaction, R, from a uniform load for the distance a and an added half-panel load, $\frac{1}{2}wp$, distant a from the right support, Thus,

$$V=\frac{wa^2}{2l}+\frac{wp}{2}\frac{a}{l},$$

or

$$V=\frac{wa}{2l}(a+p). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Designating the number of panels in l and a by N and n, respectively, Eq. (16) becomes

$$V=\frac{n(n+1)}{2N}wp. \quad . \; . \; . \; . \; . \checkmark . \quad (17)$$

The stresses, S, in the diagonals are obtained in the usual way as the products of V and h'/h, i.e.,

$$S=\frac{n(n+1)}{2N}wph'/h. \quad . \; . \; . \checkmark . \quad (18)$$

Since the factor $wph'/2Nh$ is a constant in any given case, the stresses may be quickly evaluated by means of a slide rule. ✓

The factor $n(n+1)/2N$ in Eqs. (17) and (18) represents algebraically the 'coefficient' of the web stress (Art. 59). It might have been derived directly by considering that the coefficient for the stress in the diagonal in the $(n+1)$th panel from the right support is

$$\frac{1}{N}+\frac{2}{N}+\frac{3}{N}+\ldots+\frac{n}{N}=\frac{n(n+1)}{2N}. \quad . \; \checkmark . \quad (19)$$

The numeric values of this coefficient, for values of N from 4 to 12 are tabulated in Art. 124.

In case the number of diagonals exceeds 3, the correctness of the results may be conveniently checked by noting that their second differences should be constant. Letting the constant $wph'/2Nh=c$ and observing that for the left end panel $n=N-1$, for the next panel $n=N-2$, etc., the diagonal web stresses and their successive differences may be expressed as follows:

Panels.	Web Stresses.	First Differences.	Second Differences.
1st	$c(N-1)N$ ✓		
		$c(2N-2)$	
2d	$c(N-2)(N-1)$ ✓		$2c$
		$c(2N-4)$	
3d	$c(N-3)(N-2)$ ✓		$2c$
		$c(2N-6)$	
4th	$c(N-4)(N-3)$ ✓		$2c$
		$c(2N-8)$	
5th	$c(N-5)(N-4)$ ✓		

99. Comparison of Live-Load Bending Moments in Trusses and Beams. Figs. 23 (*a*) and (*b*) represent a truss and a beam of equal span length and under like loading. In Fig. (*a*), the loading in any panel, as BC, for example, is conveyed by the stringers in that panel to the floor beams at B and C, the corresponding stringer reactions on the floor beams and floor-beam reactions on the trusses being r_3 and r_4. The forces r_3 and r_4 are components of the resultant of the loads in panel BC. They act downward with respect to the trusses and upward with respect to the floor beams. The moment of r_3 and r_4 about any point equals the moment of the loading in panel BC about that point. In Fig. (*a*) R denotes the *total* reaction at the left end—that is to say, the reaction of the truss plus the reaction r_1 of the end stringer. If an end floor beam exists at A, r_1 is conveyed to the truss at A, increasing the reaction of the truss by that amount, but producing of course no stresses in the truss.

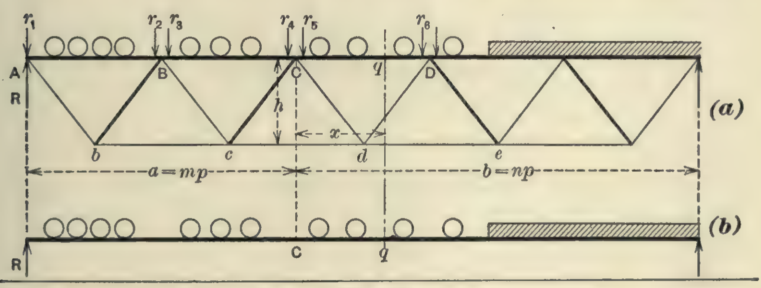


FIG. 23.

First. Considering the bending moment, M, at any joint of the loaded chord, as C, for example, and letting M_C denote the moment of the loads to the left of C about that point, it has been seen that M_C is equal to the moment of r_1, r_2, r_3 and r_4 about C. Hence

$$M = Ra - M_C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Second. This equation is seen to express also the value of the bending moment in the beam, Fig. (*b*), at a section C corresponding to joint C in the truss.

Considering now the bending moment M', at a point q in panel CD at a distance x to the right of C, it is evident from the foregoing that the moment, M_q, of R and all loads to the left of C about q is the same for the truss and the beam. For the truss,

$$M' = M_q - r_5 x. \qquad (21)$$

For the beam, letting m_q denote the moment of all loads between C and q about q, the bending moment, M_1, at q has the value

$$M_1 = M_q - m_q. \qquad (22)$$

Subtracting Eq. (21) from Eq. (22),

$$M_1 - M' = r_5 x - m_q \qquad (23)$$

which is seen to be the bending moment at q in the stringer CD.

Eqs. (20) and (23) indicate the following relations between the live-load bending moments in a truss and a beam, for equal span lengths and like loading.

1. The bending moments in a truss at the joints of the loaded chord are equal to those at corresponding sections of a beam.

2. The bending moments in a truss at the joints of the unloaded chord are less than those at corresponding sections of the beam, the differences being equal to the bending moments at corresponding sections in the stringers. If the panels in question contain no loads the bending moments in the stringers are zero, and the bending moments at corresponding sections in the truss and beam are then equal throughout that panel.

100. Chord Stresses. *First Method.* Since a vertical downward load at any point of a simple truss produces positive bending moments throughout the truss, and since the chord stresses are direct functions of the bending moments, it follows that the chord stresses become maximum throughout the truss when the uniform live load covers the entire bridge.

For a uniform load of intensity w per truss over the whole bridge, it has been seen (Art. 42) that the bending moment at joint C, Fig. 23, is

$$M = \tfrac{1}{2} wab;$$

or

$$M = \tfrac{1}{2} wp^2 mn \qquad (24)$$

The corresponding chord stress S in member cd of the unloaded chord is

$$S = \frac{wp^2}{2h} mn. \qquad \qquad (25)$$

Graphically the bending moments in a beam for a uniform load over its entire length are represented by ordinates to the parabola in Fig. 24. The bending moments in a five-panel truss are represented by ordinates to the inscribed polygon. The bending moment at a joint in the unloaded chord, Fig. 23 (*a*), as c, for example, is therefore the mean of the bending moments at joints B and C in the loaded chord. Hence after the stresses in the members of the unloaded chord have been derived by Eq. (25), the stresses in the members of the loaded chord may be found by interpolation. Thus,

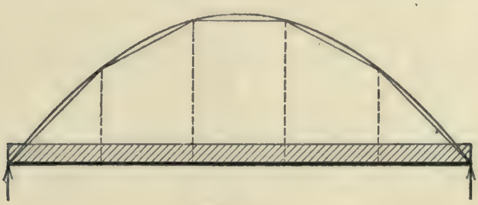


FIG. 24.

$$\text{stress } BC = \tfrac{1}{2}\,(\text{stress } bc + \text{stress } cd). \qquad (26)$$

If the moment center c for stress BC does not lie midway between B and C, then if k denotes the ratio of the horizontal distances of c and C from B,

$$\text{stress } BC = \text{stress } bc + k(\text{stress } cd - \text{stress } bc).$$

In trusses with vertical web members, such as the Pratt and Howe trusses and the triangular truss with verticals, the joints in the loaded and unloaded chords lie in the same vertical planes, so that the stresses in both chords may be found by Eq. (25).

The bending moments referred to above as the bending moments *at* the joints of the chords, as is customary in practice, are in reality the bending moments in vertical sections through these joints. The bending moments in a truss need rarely be considered at other points than the joints. As a question of statics it is evident, however, that the bending moment at any intermediate point, as q, for example, is balanced by the moments with respect to that point of the stresses in all members traversed by any section cutting q.

Second Method. The maximum live-load chord stresses may also be derived from the dead-load stresses by multiplying the latter by a common factor. If w denotes the live load in pounds per foot of bridge, and p the panel length, then for the usual case of two trusses, the live apex load, P, equals $\frac{1}{2}wp$. If W' and W_1 denote, respectively, the upper and lower dead apex loads, and S_l and S_d the live-load stress and dead-load stress in any chord member:

$$S_l = \frac{P}{W' + W_1} S_d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

provided W' and W_1 have the same coefficients in the expressions for the dead-load chord stresses, as in the Pratt truss (Art. 83), the Howe truss (Art. 86), the triangular truss with verticals (Art. 88) and the Whipple truss (Arts. 90 and 92).

In case W' and W_1 have different coefficients in the expressions for the dead-load chord stresses, the latter should be computed separately for W' and W_1. Designating the corresponding component dead-load stresses by S' and S_1, respectively:

For *deck bridges,*

$$S_l = \frac{P}{W'} S', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and for *through bridges,*

$$S_l = \frac{P}{W_1} S_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

These equations apply to the triangular truss without verticals (Art. 87), the double-triangular truss (Arts. 94–95), and the quadruple-triangular truss (Art. 96).

101. Maximum and Minimum Stresses. Some specifications require the maximum and minimum stresses in a member to be considered in proportioning the same. Where the dead and live-load stresses are always of the same character, as in the chord members of simple trusses, the dead-load stress is the minimum. In case the live load may induce in a member stresses opposite in character to the dead-load stress, three conditions may determine the minimum stress.

Let S_d = dead-load stress in the member.

S_l = greatest live-load stress in the member, opposite in character to S_d.

(1) When $S_l < S_d$, $S_{\text{min}} = S_d - S_l$.

(2) When $S_l > S_d$, and the member is so proportioned, or so connected, as to be incapable of developing stress opposite in character to S_d, $S_{\text{min}} = 0$.

(3) When $S_l > S_d$, and the member is counterbraced—i.e., capable of developing stress opposite in character to S_d—$S_{\text{min}} = S_l - S_d$. In this case S_{min} is opposite in character, and therefore in sign, to S_d.

The maximum stress, S_{max}, is the sum of the dead-load stress and the greatest live-load stress of the same character.* Since the resultant stress in a member may have any value between S_{min} and S_{max}, according to the position and magnitude of the live load, it follows that for condition (3) above, the arithmetical minimum is zero. However, since the method of proportioning members is based on the greatest possible *range* of stress, S_{min} should be found as above stated as the greatest stress opposite in character to S_{max}.

It is seen from the foregoing that in order to meet the requirements of specifications involving this method of proportioning members, it is necessary to compute for every member subject to live-load stresses opposite in character, the maximum stress of each character and then to combine each of these stresses with the dead-load stress.

102. Combination of Dead and Live-Load Stresses in Counters. The methods of proportioning bridge members will be treated in Vol. III. It is necessary, however, to consider briefly now the various ways in which the dead and live-load stresses are usually combined in proportioning counters.

As explained in Art. 74, in steel trusses a counter is usually a flexible member incapable of resisting compression, whose stress under dead loading only, is zero. If counters are provided the main diagonals in panels containing counters are not counterbraced, and these members are therefore incapable of developing resultant compressive stresses. Since the dead-load stresses in these members are tensile they are, however, capable of resisting live-load compressive stresses up to the limits at which the dead-load tensile stresses are neutralized. Under ideal adjustment the counter will not come into play until the dead-load tensile stress in the main diagonal in the same panel has been completely exhausted (Art. 74); it will then develop the required additional stress in tension. When the counter thus acts in tension, the stress in the main diagonal is zero, so that it may be regarded as having no existence. It follows therefore that under these conditions the resultant tensile stress in the counter is the difference between the tensile live-load stress, S_l, and the compressive dead-load stress, S_d, in that member on the assumption that the main diagonal does not exist. If S_l denotes the live-load stress in the counter when the load occupies the position for maximum tension in that member, the resultant stress, S_c, in the counter is

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* In some rare instances this value of S_{max} is *less* than S_{min} for condition (3) above. The latter value should then be called S_{max} and the former S_{min}.

$$S_c = S_l - S_d. \qquad (30)$$

Since, however, the ideal adjustment referred to above cannot be maintained, and since, moreover, rapidly moving live loads produce momentarily greater stresses than static loads in the same position, it is customary to recognize these elements by one of the following methods:

First Method. By this method the live-load stress in every member is transformed into an 'equivalent' dead-load stress by adding a certain percentage, k, determined by various formulas which will not be considered at present, to allow for 'impact.' This term signifies the resultant effect on the stress of all influences due to the circumstance that the load is in motion. For counters a reduction factor, usually $\frac{2}{3}$, is also applied to the dead-load stress, which serves to provide a safety margin to cover (1) future increased loading, (2) the effect on the counters of an occasional load of extraordinary weight, and (3) excessive initial tension in the counters from screwing them up too tightly. Thus

$$S_c = (1+k)S_l - \tfrac{2}{3}S_d. \qquad (31)$$

The sectional area, a, of the counter is then found by dividing S_c by the allowable unit stress, s_d, for dead-load stresses, whence

$$a = [(1+k)S_l - \tfrac{2}{3}S_d] \div s_d. \qquad (32)$$

Second Method. By this method the sectional area of each member is found by dividing its dead-load stress by a certain unit stress and its live-load stress by one-half that unit stress, and adding the resulting required sectional areas. This is equivalent to adding 100 per cent to all live-load stresses for impact. For counters no further reduction factor is usually applied to the dead-load stress. Thus

$$a=(2S_l-S_d)\div s_d.$$

If s_l denotes the allowable unit stress for live-load stresses, $s_l=\frac{1}{2}s_d$, and

$$a=(S_l-\tfrac{1}{2}S_d)\div s_l. \quad \dots\dots \quad (33)$$

This equation may be interpreted as signifying that the dead-load stress is reduced to an 'equivalent' live-load stress by the factor $\frac{1}{2}$, then combined with the live-load stress, and this value—i.e., the resultant 'equivalent' live-load stress—is divided by the allowable unit stress for live-load stresses to find the required sectional area of the member.

Third Method. By this method, already referred to in the preceding article, the allowable unit stress is made dependent on the range of stress to which the member is subject, according to various formulas which will not be considered at present. These formulas are so devised, however, that if s_d denotes the allowable unit stress in a member subject to dead-load stress only, and s_l the allowable unit stress in a member in which the stress may range from zero to a higher value, as in a counter, $s_l=\frac{1}{2}s_d$. For counters this method therefore leads to the same result as the second method, Eq. (33).

It is important to observe that in determining the minimum and maximum stress in each member by this method no addition is made to the live-load stresses for impact. As the range of stress increases, the allowable unit stress is reduced by the formulas; whereas by the first and second methods the live-load stresses are increased for impact, but the unit stresses are constant. In a general way the effect of these different modes of procedure on the sectional area of the member is similar.

In ordinary highway bridges not subject to heavy electric-car traffic, the maximum live-load stresses in truss members are developed less frequently and less suddenly than in railway bridges. Under such circumstances it is not customary to apply a reduction factor to S_d as in Eq. (33). The sectional area of the counter is then found by the formula:

$$a=(S_l-S_d)\div s_l. \qquad (34)$$

The first of these three methods, which is the one prescribed in the A. R. E. Specifications,* may be regarded as the most rational. It involves a variable coefficient for impact, *k*, whose value is of interest only in connection with the proportioning of the members, a subject which will not be further treated in this volume. After the live-load stresses have been determined by methods which will now be considered, proper percentages *k* may be added for impact to meet the requirements of any particular specifications. Since this factor, *k*, is, however, a variable, the combination of dead and live-load stresses for counters in all examples in this volume will be made for convenience according to Eq. (33) or (34).

103. Pratt Truss. Referring to Fig. 25, let w denote the uniform live load per unit of length, per truss. Then wp equals the maximum live apex load. A load, wp, at any apex produces a positive shear, equal to the reaction at the left support, in every panel to the left of the loaded apex. Hence, for maximum positive shear in any panel, assume a load wp applied at every apex to the right of the panel. Positive shears induce tension in all diagonals sloping downward to the right, Fig. (*a*), and compression in those sloping upward to the right, Fig. (*b*). In the analysis of web stresses, it is sufficient to consider positive shears only, since the effect of the negative shear on any diagonal, as Bc, for example, from a load at b, is the same as the effect of the positive shear from a load at h on the corresponding member Hg, in the other half of the truss.

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* These specifications, issued by the American Railway Engineering Association, under the title "General Specifications for Steel Railway Bridges," will be used throughout this work, and will be referred to as the "A. R. E. Specifications." These specifications were prepared and are revised periodically by a large and representative committee of bridge engineers, known as the Committee on Iron and Steel Structures.

(*a*) *Main Diagonals.* Let it be assumed at first that the counters, represented by broken lines, do not exist, and that the main diagonals near the middle of the truss, in which the dead-load tensile stresses may be more than neutralized by the live-load compressive stresses, are counterbraced and thus enabled to resist compression. For maximum tension in any main diagonal, as *Cd*, for example, a full panel load, *wp*, should be applied at every apex from *d*, inclusive, to the right. For maximum compression in that member panel loads *wp* should be applied at joints *b* and *c* to the left; or, since it is more convenient to consider positive shears only, joints *g* and *h* may be assumed to be loaded for maximum compression in the corresponding member *Gf* in the other half of the truss.

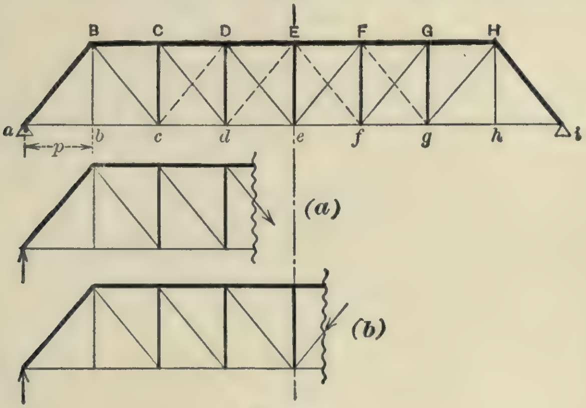


FIG. 25.

(*b*) *Counters.* In considering the maximum stress in a counter, as *Ef*, for example, assume that the main diagonal *Fe* in that panel does not exist (Art. 102). For maximum live-load tension, S_l, in *Ef*, joints *f*, *g* and *h* should be fully loaded. Let S_d equal the dead-load compressive stress in *Ef* if *Fe* does not exist. This is equal to the dead-load tensile stress in *Fe*. Then as explained in Art. 102, Second Method, the resultant maximum tensile stress in the counter is

$$S_c = S_l - \tfrac{1}{2}S_d.$$

If $S_l < \tfrac{1}{2}S_d$ no counter is needed in that panel. In that case it is considered that the compressive 'counter stress,' S_l, in the main diagonal *Fe* will not be sufficient to neutralize momentarily the dead-load tensile stress S_d in that member.

(*c*) *Verticals.* The maximum live-load stress in any intermediate post is equal to the vertical component of the live-load stress in the diagonal intersecting that post in the *unloaded* chord. It is therefore equal to the maximum live-load shear in the adjacent panel toward the farther end of the truss. Thus stress *Cc* is maximum when joints *d* to *h*, inclusive, are fully loaded, and it is equal to the shear in panel *cd*.

The minimum stress in the posts may be less than the dead-load stress, as will be explained in the example following.

(*d*) *Chords.* The live-load chord stresses are found as explained in Art. 100.

Example. Fig. 26 represents a highway bridge, 128 ft. long, center to center of end supports. The live load per foot of bridge is 1500 lbs. Find the live-load web stresses by the conventional method.

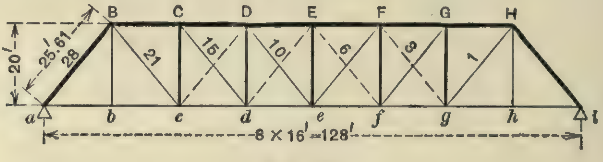


FIG. 26.

Solution. Panel loads and stresses will be expressed in thousands of pounds. Panel load, $wp = \frac{1}{2}(1.5 \times 16) = 12$.

In Eq. (18), Art. 98.

$$\frac{wp}{2N} = \frac{12}{2 \times 8} = 0.75.$$

Diagonals. From Eq. (18), Art. 98:

		Shears	$\times h'/h$	
	Stress $aB =$	$0.75 \times 7 \times 8 (=42)$	$\times \frac{25.61}{20}$	$= (-)53.8$
	" $Bc =$	" $\times 6 \times 7 (=31.5)$	$\times$ "	$= (+)40.3$
	" $Cd =$	" $\times 5 \times 6 (=22.5)$	$\times$ "	$= (+)28.8$
	" $De =$	" $\times 4 \times 5 (=15)$	$\times$ "	$= (+)19.2$
Counters	" $Ef =$	" $\times 3 \times 4 (= 9)$	$\times$ "	$= (+)11.5$
	" $Fg =$	" $\times 2 \times 3 (= 4.5)$	$\times$ "	$= (+)\ 5.8$
	Counter stress $Bc =$	" $\times 1 \times 2 (= 1.5)$	$\times$ "	$= (-)\ 1.9$

The stresses in counters Ef and Fg should be diminished by the dead-load tensile stresses in the main diagonals Fe and Gf, respectively (Eq. (34), Art. 102).

Counter stress Bc is of interest only if the minimum stress in that member is required by the specifications for proportioning the same (Art. 101). The minimum stress is then found by subtracting the counter stress from the dead-load stress. The minimum stress in counters and in the main diagonals in panels containing counters is zero. The minimum stress in aB is its dead-load stress.

Verticals. The stresses in the intermediate vertical posts Cc, Dd and Ee are obtained from the shears tabulated above, viz:

$$\begin{aligned}\text{Stress } Cc &= -22.5,\\ \text{Stress } Dd &= -15,\\ \text{Stress } Ee &= -\ 9.\end{aligned}$$

The stress in the end suspender Bb is equal to the apex load at b, i.e.,

$$\text{Stress } Bb = +12.$$

When joints g and h are loaded for maximum stress Fg, stress Gf is zero. The simultaneous stress in Gg is therefore the dead-load concentration at G, which is the minimum stress in that member. When joints f, g and h are loaded, for maximum stress Ef, stress Fe is zero. If, for that loading, the resultant dead and live shear in panel fg is negative, counter Fg is not in action, and the minimum stress in Ff is then equal to the dead-load concentration at F. If the shear in panel fg is positive, the simultaneous stress in Ff is equal to the dead-load concentration at F plus the vertical component of stress Fg. This stress may be greater or less than the dead-load stress in Ff. The minimum stress in the central post Ee and in the hip vertical Bb is the dead-load stress.

The stresses in the diagonals may be expressed even more simply by deriving the coefficient $n(n+1)/2$ in Eq. (18), Art. 98, directly by beginning with 1 and adding successive integers, as in Eq. (19), Art. 98. Thus, in Eq. (18),

$$\frac{wp}{N} = \frac{12}{8} = 1.5.$$

$$\begin{array}{lrlcll}
 & & & \text{Coeff's} & \text{Shears} \times h'/h & \\
 & \text{Counter stress } Bc & =[1.5\times & 1 & =\ 1.5]\times\dfrac{25.61}{20} & =(-)\ 1.9 \\
\text{Counters}\left\{\begin{array}{l}\\ \\ \end{array}\right. & \begin{array}{r}\text{Stress } Fg\\ Ef\end{array} & \begin{array}{l}=[\text{“}\times\\ =[\text{“}\times\end{array} & \begin{array}{c}(1+2=\ 3)\\ (3+3=\ 6)\end{array} & \begin{array}{l}=\ 4.5]\times\ \text{“}\\ =\ 9\ \ ]\times\ \text{“}\end{array} & \begin{array}{l}=(+)\ 5.8\\ =(+)11.5\end{array} \\
 & De & =[\text{“}\times & (6+4=10) & =15\ \ ]\times\ \text{“} & =(+)19.2 \\
 & Cd & =[\text{“}\times & (10+5=15) & =22.5]\times\ \text{“} & =(+)28.8 \\
 & \text{‘}\quad Bc & =[\text{“}\times & (15+6=21) & =31.5]\times\ \text{“} & =(+)40.3 \\
 & \text{“}\quad aB & =[\text{“}\times & (21+7=28) & =42\ \ ]\times\ \text{“} & =(-)53.8
\end{array}$$

The coefficient for each diagonal is indicated in Fig. 26.

The chord stresses may be found as explained in Art. 100.

104. Combination of Dead and Live-Load Stresses as Influenced by Counters. Simultaneous Stresses. The resultant stress in certain members from dead and live loading depends upon whether the main diagonal or the counter in each of one or more panels is in action. When the counter acts, the stress in the main diagonal in the same panel is zero and that member may be regarded as not existing. Thus, in general, *the stress in a counter is the resultant stress in that member from dead and live loading if the main diagonal in the panel is assumed to be removed.* But when the dead-load web stress is borne by the counter instead of the main diagonal, the dead-load stress in certain other members is affected by this change. The resultant stress (dead plus live-load stress) in such members can be correctly obtained only by combining the live-load stress and the *simultaneous* dead-load stress, when the main diagonal or the counter is in action, as the case may be. This principle is frequently of importance in the analysis of stresses (as in the determination of the minimum stresses in the posts in the example in the preceding article), and sometimes in the determination of the maximum stress in members in which the resultant stress may become maximum when the counters are in action, although such instances are comparatively rare. A case of this kind will be discussed in the next article. The application of this principle to roof trusses was explained in Art. 74.

The central vertical in a deck Pratt truss will serve to illustrate the application of the principle referred to. Under the dead loading shown in Fig. 27 (*a*), stress $Ee = -W'$ and the stress in the dotted counters is zero. Let wp denote the live apex load, and let such loads be applied at B, C and D, Fig. (*b*.) The live-load shear in panel DE is then equal to $+\frac{3}{4}wp$, and if this exceeds the dead-load shear, $-\frac{1}{2}(W' + W_1)$, in that panel (which will always be true for spans of moderate length), the resultant shear is positive and counter Ed is then in action. Since the stress De is then zero, section 1 may be taken and considering the live and dead loads and reactions to the left of the section,

$$\text{Resultant stress } Ee = -(\tfrac{3}{4}wp + \tfrac{1}{2}W' - \tfrac{1}{2}W_1).$$

In a deck truss $W' > W_1$; therefore the simultaneous dead-load stress $\frac{1}{2}(W' - W_1)$ in Ee is compressive, but relatively much less than W', the normal dead-load compressive stress in *Ee* when the counter *Ed* is not in action. The above resultant stress *Ee* is the stress in the middle post when the loading occupies the position for maximum stress in counter *Ed*. The conditions for maximum stress *Ee* will now be considered.

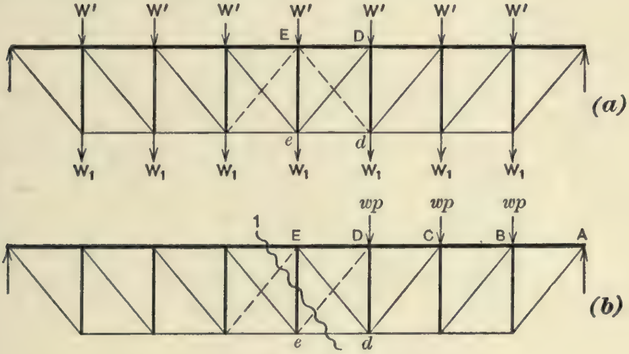


FIG. 27.

105. Maximum Stress in the Middle Vertical of a Truss.

The analysis of the conditions for maximum stress in the middle post of a deck Pratt truss or a through Howe truss involves the principle discussed in the preceding article.

Referring to Fig. 27, let the conditions for maximum stress in *Ee* be considered. Assume a load *wp* at *E* in Fig. (*b*) in addition to like loads at *B*, *C* and *D*. The live-load shear in panel *DE* is now $1\frac{1}{4}wp - wp = +\frac{1}{4}wp$, whereas the dead-load shear in that panel, Fig. (*a*) is $-\frac{1}{2}(W' + W_1)$. If $\frac{1}{4}wp > \frac{1}{2}(W' + W_1)$ the resultant shear is positive and counter *Ed* is in action. In this case

$$\text{Resultant stress } Ee = -(1\tfrac{1}{4}wp + \tfrac{1}{2}W' - \tfrac{1}{2}W_1), \quad . \quad . \quad (35)$$

and this is then the maximum compressive stress in the central post, since a load at any apex to the left of *E* will induce tension in that member.

If, however, $\frac{1}{4}wp < \frac{1}{2}(W' + W_1)$, the resultant shear in panel *DE* is negative, and *De* acts instead of *Ed*. Since the shear in the panel to the left of *E* is positive, neither of the counters meeting at *E* is in action, and stress *Ee* is therefore equal to the total load at *E*—i.e.,

$$\text{Resultant stress } Ee = -(wp + W'). \quad . \quad . \quad . \quad . \quad (36)$$

Comparing Eqs. (35) and (36) it is seen that the stress from the former is greater when $\frac{1}{4}wp > \frac{1}{2}(W' + W_1)$, which is also the condition premised in the derivation of Eq. (35). Thus, in general, for an eight-panel truss, when the intensity of the live load is more than twice as great as that of the dead load, Eq. (35) governs, and otherwise, Eq. (36). The resultant stress Ee in such a case is frequently given as $1\frac{1}{4}wp + W'$, which is seen to be incorrect.

In a truss of six panels loaded from the right end up to the center, the live shear in the panel to the right of the center is zero, and in a four-panel truss similarly loaded it is negative. Hence the maximum resultant stress in the central post should be found in both cases by Eq. (36).

For trusses of 10, 12 and 14 panels the factor $1\frac{1}{4}$ in Eq. (35) becomes $1\frac{1}{2}$, $1\frac{3}{4}$ and 2, respectively, and Eq. (35) thus modified will govern when $\frac{1}{2}wp$, $\frac{3}{4}wp$ and wp are respectively greater than $\frac{1}{2}(W' + W_1)$. These conditions will be fulfilled under all ordinary conditions.

106. Howe Truss. The points of difference between the Howe and Pratt trusses have been considered in Art. 86. The maximum live-load stresses in the diagonals of a Howe truss are equal to those in a Pratt truss of like dimensions under like loading, but they are compressive instead of tensile. The stresses in the verticals are, however, not only opposite in character, but different in magnitude.

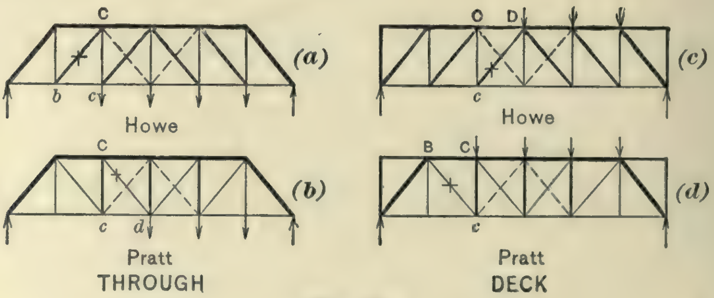


FIG. 28.

Referring to Figs. 28 (*a*), (*b*), (*c*) and (*d*), it is seen that the live-load stress in Cc, for example, is always equal to the vertical component of the stress in the diagonal which meets that member in the *unloaded* chord, and that stress Cc therefore becomes maximum when the load occupies the position for maximum stress in that diagonal. This diagonal is distinguished in the several figures by cross marks, and the loaded apexes for maximum stress in this diagonal are distinguished by arrows. From a consideration of these figures it is seen that the following relations exist between the maximum live-load stresses in corresponding verticals, for trusses of like dimensions and live loading of like intensity:

1. The stresses in the through Howe and deck Pratt trusses are equal.
2. The stresses in the deck Howe and the through Pratt trusses are equal.
3. The stresses in the through Howe and deck Pratt trusses are greater than those in the deck Howe and through Pratt trusses.

107. Triangular Truss. In the triangular truss, the tension diagonals are counterbraced whenever the live-load counter stress more than neutralizes the dead-load tensile stress.

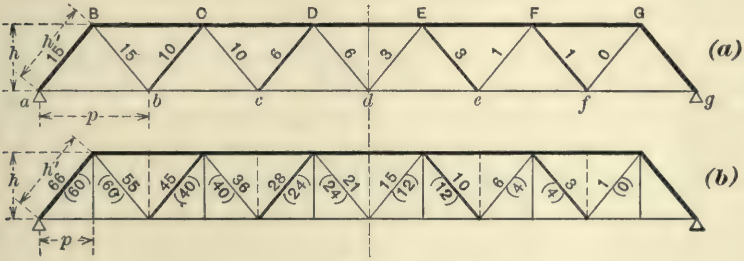


FIG. 29.

(*a*) *Truss without Verticals.* The coefficients for the web stresses for maximum live-load shear in each panel, found by adding successive integers, are shown in Fig. 29 (*a*). The coefficient for Cc, for example, equals $1+2+3+4=10$. If w denotes the intensity of the uniform live load per truss, the maximum live-load stresses are found by multiplying the coefficients to the left of the center line by $\frac{1}{6}wph'/h$, and the maximum counter stresses are derived similarly from the coefficients to the right of the center line.

(*b*) *Truss with Verticals.* The presence or absence of the dotted verticals, Fig. 29 (*b*), does not affect the live-load stresses in the truss, inasmuch as the live-load stresses in these members are zero. Since the number of panels is doubled by the full-line verticals, the coefficients in Fig. (*b*) for maximum live-load and counter stresses must be multiplied by $\frac{1}{12}wph'/h$, in which p is one-half as great as in Fig. (*a*). The coefficients for the latter, multiplied by four, are indicated in parentheses in Fig. (*b*) to show clearly the effect of the verticals on the stresses in the diagonals.

(*c*) *Chord Stresses.* The chord stresses may be found as explained in Art. 100.

108. Whipple Truss. (*a*) *Even Number of Panels.* It was shown (Art. 90) that the stresses in a Whipple truss with an even number of panels are statically determinate for counterbalanced loading. For a live load of uniform intensity, the stresses in all chord members become maximum when the load extends over the entire bridge. For the usual condition of equal panels, the live loads are then counterbalanced, so that the live-load chord stresses are statically determinate. These stresses may therefore be derived as explained in Art. 100.

With the live load in position for maximum stress in any intermediate web member, the loads are not counterbalanced, so that the counters are necessarily brought into play. The stresses then become statically indeterminate, which may be shown as follows:

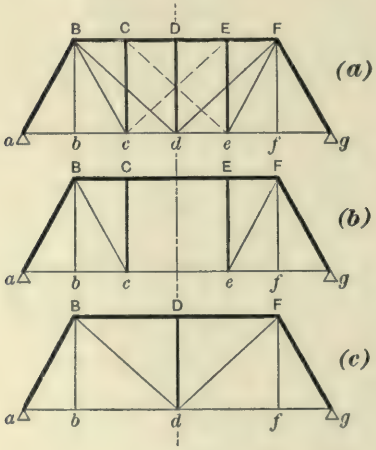


Fig. 30.

In the six-panel truss Fig. 30 (*a*), the counters may be disregarded for counterbalanced loading and the truss may be divided into the two component trusses shown in Figs. (*b*) and (*c*). Disregarding counters, there are 21 members and 12 joints in the composite truss, Fig. (*a*), so that the criterion for redundancy, Art. 47, is satisfied. For unbalanced loading, the upper truss, Fig. (*b*), is not stable, however, without counters. For a load at *b* or *c*, counter *Ec* will be brought into play, and for a load at *e* or *f*, counter *Ce* will act. The number of members in the truss is thus increased to 22, and the criterion for redundancy is no longer met. Again, if a load is applied at the foot of one of the hip verticals, for example at b, that part of the load transferred to the right end support may be carried either by web system (b) with counter Ec acting, or by web system (c). Its division between the two web systems is statically indeterminate, but the error involved in the usual assumption that one-half the shear is carried by each system is relatively small. The conventional method for the determination of the live-load stresses will be illustrated in detail for a truss with an odd number of panels.

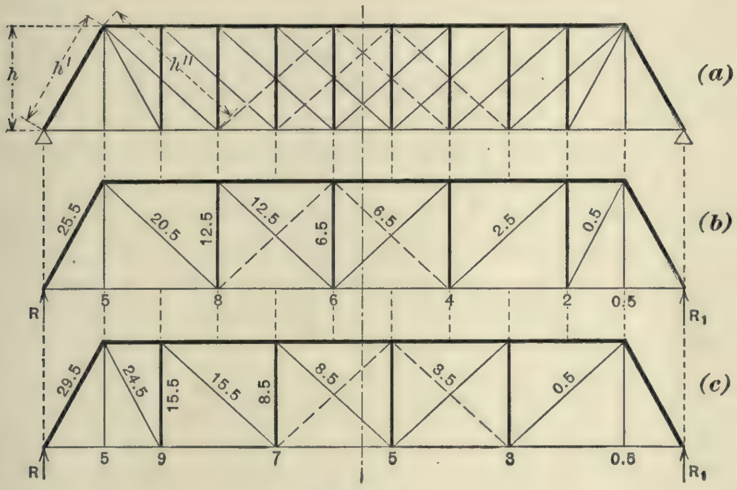


FIG. 31.

(b) *Truss with Odd Number of Panels.* Assuming the truss of 11 panels shown in Fig. 31 (a), for which the dead-load stresses were derived in Art. 92, let it be divided into the two component trusses, Figs. (b) and (c). Let each of these trusses be considered to carry independently the loads applied to its own joints, except those at the lower ends of the hip verticals, which will be assumed as divided equally between the two web systems. If w denotes the intensity of the uniform live load for one truss, the values indicated at the lower-chord joints are factors which, when multiplied by $\frac{1}{11}wp$, will give the left end reaction, R, from a load at any joint—i.e., the positive shear in all panels to the left of that joint. The coefficient for any diagonal in Figs. (*a*) and (*b*) is obtained by the summation of the factors to the right of the corresponding panel. Thus, in Fig. (*b*), $12.5=0.5+2+4+6$. The two component trusses are seen to be alike, if one is turned end for end. Hence the coefficient for the counter stress in any main diagonal has the value shown on the corresponding member in the right half of the other truss. The coefficient for the inclined end post is equal to the sum of the coefficients for that member in Figs. (*b*) and (*c*), i.e., $25.5+29.5=55$. The coefficient for any vertical post is equal to that for the diagonal meeting that post in the unloaded chord in the left half of each truss.

The maximum live-load stress for any vertical post is the product of its coefficient and $\frac{1}{11}wp$. For the diagonals each coefficient is to be multiplied by $\frac{1}{11}wph'/h$ or $\frac{1}{11}wph''/h$ according to the slope of the member. The resulting stresses for the counters should be diminished by S_d, the dead-load stress in the main diagonal in the same double panel, or by S_d multiplied by a suitable reduction factor (Art. 102). The maximum live-load stress in the hip verticals is $+wp$.

Problem. Find the maximum live-load web stresses in the Whipple truss with an even number of panels shown in Fig. 10 (*a*), Art. 90, for a live-load of intensity w for one truss.

109. Double and Quadruple Triangular Trusses. The maximum live-load web stresses in the double-triangular truss, Fig. 16 (*a*), Art. 93, and in the quadruple triangular truss, Fig. 19 (*a*), Art. 96, may be found by the method explained in the preceding article for the Whipple truss. Each component truss is assumed to carry independently the loads applied at its own joints. Full panel loads, wp, are assumed so placed as to produce maximum positive shear in each panel in turn. The coefficients for maximum stress and counter stress are then derived, and the stresses are obtained by multiplying these coefficients by the usual factors. The tension diagonals are counterbraced whenever the live-load counter stress more than neutralizes the dead-load tensile stress. While, for reasons stated in Arts. 93 and 96, the stresses in these trusses are indeterminate (except those in the double-triangular truss, with an even number of panels, for counterbalanced loads), the errors involved in the live-load stresses by the above conventional method are relatively small, and the method meets practical requirements satisfactorily.

The chord stresses may be found as explained in Art. 100.

Problem 1. Find the maximum live-load web stresses in the double-triangular truss, Fig. 16 (*a*), Art. 93, for a live load of intensity w for one truss, letting p=panel length, h=height of truss, h'=length of diagonals.

Problem 2. Find the maximum live-load web stresses in the quadruple triangular truss, Fig. 19 (*a*), Art. 96, for a live load of intensity w for one truss.

D. STRESSES FROM CONCENTRATED LIVE LOADS

110. General Considerations. In the determination of the live-load stresses in the trusses of highway bridges and in the principal truss members of very long railroad bridges, it is usually assumed that the live-load is uniformly distributed. The live-load stresses in railroad bridges for spans up to about 500 ft. are generally computed by assuming the live load to consist of a series of axle loads representing two locomotives coupled, followed by a uniform train load (Art. 38). The position which such a load system must occupy to produce the maximum stress or counter stress in a given member, and the stresses themselves, may be found by various general methods, of which the *algebraic* method—also known as the *analytic* method—as distinguished from various graphic methods, is usually regarded by practiced computers as the most direct and convenient for ordinary bridge trusses. The application of this method to the most common types of trusses with horizontal chords will be considered in this chapter. The numeric evaluation of the stresses from a series of concentrated loads may be performed most rapidly and conveniently with the aid of a moment table (Art. 39) and a slide rule.

111. Pratt Truss. (*a*) *Diagonals.* Referring to Fig. 32, the stress in any diagonal, as *Cd*, for example, becomes maximum when the load occupies the position in which it will produce the greatest shear in the corresponding panel, *cd*. In this and succeeding figures W, W_1, etc., denote the sums of the loads included in the corresponding braces in the figures; and x, x_1, etc., the distances from their respective centers of gravity to a fixed point. Thus in Fig. 32, Wx denotes the moment of the total load on the truss with respect to the right end support. The live load will be assumed in general *to advance from the right,* producing maximum positive shears in successive panels as it advances. Moment tables are usually prepared on this assumption.

Any load to the left of c will produce negative shear in panel cd. It will therefore be assumed at first that no loads exist on that segment of the truss when the loading is placed in the position for maximum stress in Cd. Let the loading be assumed to occupy some such position as shown in the figure, with a wheel load at d. Let r equal the forward apex load (floor-beam reaction) at c, and let V equal the shear in panel cd. Then

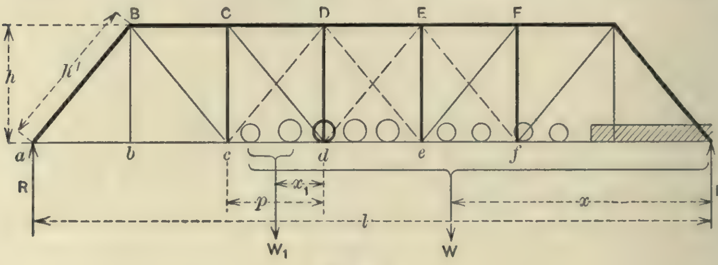


FIG. 32.

$$V = R - r, \quad \quad \quad (37)$$

or

$$V = \frac{Wx}{l} - \frac{W_1 x_1}{p}. \quad \quad \quad (38)$$

To find the condition for maximum V, let the first derivative dV/dx equal zero, noting that $dx = dx_1$, since any differential change in x from a differential movement of the load will be accompanied by an equal differential change in x_1. Then

$$\frac{dV}{dx} = \frac{W}{l} - \frac{W_1}{p} = 0,$$

or

$$W = \frac{l}{p} W_1. \quad \quad \quad (39)$$

For the usual case of equal panels, if N denotes the number of panels in the truss,

$$W = NW_1. \quad \quad \quad (40)$$

Eqs. (39) and (40) express the criterion for maximum shear in any panel, and therefore for maximum stress in any diagonal, W_1 denoting the total load in the corresponding panel. From Eq. (39) $W/l=W_1/p$, which shows that *the stress in any diagonal becomes maximum when the average live load per unit of length for the corresponding panel is equal to that for the whole truss.* In seeking the position for maximum positive shear in any panel, it is to be noted that since W decreases for successive panels to the right, W_1 must be allowed to decrease proportionately to satisfy the criterion. The latter is satisfied if, when the load at d is assumed to act at a differential distance first to the right, and then to the left of d, the second member of Eqs. (39) and (40) is first less, and then greater, than the first member; for mathematically this implies a transition through the condition of equality. This is equivalent to assuming the load at d to be divided into two parts, the one applied immediately to the left, the other immediately to the right of that point, in such proportions as to satisfy the criterion. The mathematical significance of the criterion for maximum will be further discussed in the next article.

The load which when placed at d satisfies the criterion, is called the 'critical load,' and the apex d the 'critical point,' for diagonal Cd. The critical load must be found by trial. For web stresses the critical load is always one of the heavy drivers of the first engine. Under certain conditions to be considered later (Art. 117), two successive loads may each satisfy the criterion for maximum web stress. In that case the shear must be found for each position and the maximum value used in computing the stress. The critical point is located at that apex at which an assumed differential movement of the critical load will affect the value of W_1 in the criterion.

In rare cases of very short panels or unusual load systems, one or more loads may be found to the left of the panel in question when the loading occupies the position satisfying the criterion. If W_2 denotes the sum of any loads to the left of c, Eq. (37) becomes

$$V=R-r-W_2.$$

Since W_2 is a constant, it will disappear in differentiation, so that the possible presence of loads to the left of the panel in which the maximum shear is sought will not affect the criterion.

When the loading occupies the position for maximum shear in the end panel, *ab*, the shear, V, in that panel equals R instead of $R-r$ as for the other panels. In this case $r(=W_1x_1/p)$ is the abutment reaction of the stringer in the end panel. But since now $Wx/l=R+W_1x_1/p$,

$$V=R=\frac{Wx}{l}-\frac{W_1x_1}{p},$$

as before, so that Eq. (38) is perfectly general.

After the position of the load for maximum stress in a given diagonal has been found, the shear in the panel may be computed by Eq. (37), in which R and r may be readily derived with the aid of a moment table (Art. 39). The stress in the diagonal is then Vh'/h.

The maximum stress in the counters is found in precisely the same way as that in the main diagonals. Thus, for counter *Ef* the critical wheel is placed at *f*, for the load advancing from the right. The resulting stress should be reduced by S_d, the dead-load tensile stress in *Fe* (which is also the dead-load compressive stress in *Ef* when *Fe* is not in action), multiplied by a suitable reduction factor (Art. 102).

(*b*) *Verticals.* The maximum live-load stress in any vertical post is equal to the maximum live-load shear in the adjacent panel toward the farther end of the truss (Art. 103).

The stress in the hip vertical, *Bb*, becomes maximum when the loading is placed in position for maximum load on the floor beam at *b* (Art. 45), and it is equal to that load.

(*c*) *Chord Stresses.* In the truss shown in Fig. 33, let it be required to find the maximum live-load stress in any chord member, as *CD*, for example. Taking the section 1, the moment center is located at *d*, the intersection of the other two members severed.

In taking a section through a truss it is important to observe that the floor system is connected to the truss only at panel points, so that the live loads can be communicated to the truss only at such points. It is therefore immaterial whether the section is assumed to the left of the loads in the panel, or as in the figure, or in any other position relative to these loads. If M denotes the bending moment at *d*, stress $CD=M/h$. Stress *CD* therefore becomes maximum when the loading occupies the position for maximum M. Since a load anywhere on the truss produces positive bending moments throughout the truss, for maximum M the loading must be distributed over the entire bridge, as shown in the figure. The exact position for maximum M will now be determined.

Assuming one of the loads placed at the critical point d,

$$M=\frac{Wx}{l}a-W_1x_1. \qquad (41)$$

For M_{max},

$$\frac{dM}{dx}=\frac{Wa}{l}-W_1=0,$$

or

$$W=\frac{l}{a}W_1. \qquad (42)$$

This criterion is seen to be the same as that derived in Art. 42, for the maximum bending moment at any section of a beam.

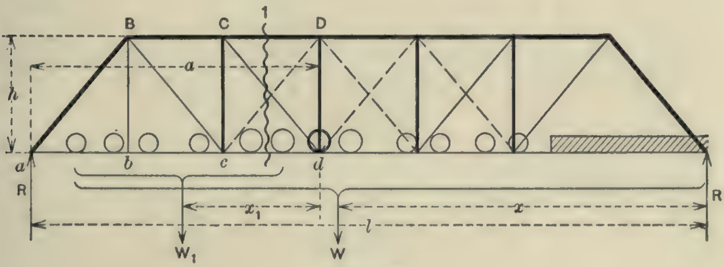


FIG. 33.

For the usual case of equal panels, letting n denote the number of panels in a, and N the total number in the truss, Eq. (42) becomes

$$W=\frac{N}{n}W_1. \qquad (43)$$

The position of loading which satisfies this criterion must be found by trial. For chord stresses several positions may usually be found to meet the requirements. For short spans and for chord members near the ends of longer spans, the stresses are usually maximum when the criterion is met by placing one of the heavy driving-wheel loads at the critical point. Since the differences in the stresses for different judiciously chosen positions which satisfy the criterion are relatively small, more than one computation is seldom made. This subject will be treated more critically hereafter (Art. 118).

From Eq. (42), $W/l = W_1/a$, which shows that *the chord stress becomes maximum when the average load per unit of length for the distance from the moment center to the support (either support) equals that for the whole truss.* For a uniform live load that condition is met when the load extends over the entire bridge, which is evidently the condition for maximum stress in every chord member.

After the position for maximum chord stress has been determined, M may be readily computed from Eq. (41) with the aid of a moment table. The chord stress is then M/h.

112. Mathematical Significance of the Criterion for Maximum. Eq. (38), Art. 111, for shear, and Eq. (41), for bending moment may each be written:

$$\mu = cWx - c_1W_1x_1,$$

in which c and c_1 are constants.

Referring to Figs. 32 and 33, and calling the critical load w_1:

If the loading is moved a differential to the right:

$$d\mu = -cWdx + c_1W_1dx. \quad . \quad . \quad . \quad . \quad (44)$$

If the loading is moved a differential to the left:

$$d\mu = cWdx - c_1(W_1 + w_1)dx. \quad . \quad . \quad . \quad (45)$$

In Eq. (44), the change, $d\mu$, in the function μ is negative when

$$c_1W_1 < cW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

In Eq. (45), the change, $d\mu$, in the function μ is negative when

$$c_1(W_1 + w_1) > cW. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

When conditions (46) and (47) are both fulfilled, the criterion for maximum is satisfied. It is seen that a differential movement in *either* direction produces then a negative change in the function μ, as should be the case for the position for maximum.

Special Case. It is found, not infrequently, especially if the panel length is in even feet, that when the loading occupies a position which satisfies the criterion for maximum, a differential movement will affect the value of W as well as W_1

in Eqs. (39), (40), (42) and (43) of Art. 111, owing to the fact that some wheel load will happen to be directly above one of the end supports. For loads advancing from the right, that usually occurs at the right support, although for bending moments it may also occur at the left support. In such cases the criterion for maximum must be interpreted in the light of these special conditions.

Let it be assumed that when the loading occupies such a position a wheel load, w, is at the right support, Fig. 34. If the loading is now moved a differential to the left:

$$d\mu = c(W+w)dx - c_1(W_1+w_1)dx. \quad . \quad . \quad . \quad . \qquad (48)$$

This change $d\mu$ in the function μ is negative when

$$c_1(W_1+w_1) > c(W+w). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

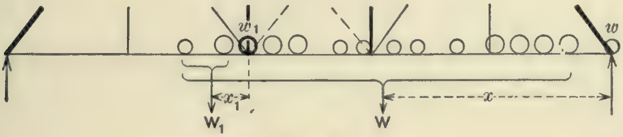


FIG. 34.

When conditions (46) and (49) are met, the criterion for maximum is satisfied. It may happen, however, that conditions (46) and (47) are met, but that condition (49) is not fulfilled. This indicates that $d\mu$ in Eq. (48) is positive, and that a movement of the loading to the left will therefore increase μ. In that case the loading must be advanced to the left until the next wheel to the right of w_1 is at the critical point and that position must then be tested by the criterion. (See Art. 113, Example 2, and Art. 115.)

113. Application of Moment Table to Finding Position of Loading to Satisfy Criterion for Maximum Stress. In seeking the position of loading to satisfy any criterion for maximum stress, lay off the span length on a straight line to the same scale as the wheel spacing in the moment table and indicate the panel divisions, as well as the distance of each apex from the right support, as shown in Fig. 35 for a 150-ft. span of six equal panels. Apply the upper edge of the moment table to the line representing the span, with some wheel at the critical point for the member under consideration. Observe by inspection the number of loads on the whole span, or any desired segment of the span or panel, and read the sum of such loads from the moment table, as explained in Art. 39. The distance covered by the uniform train load may be found by subtracting the distance from the critical wheel to the head of the uniform load (read from the table) from the distance from the critical wheel to the right end reaction.

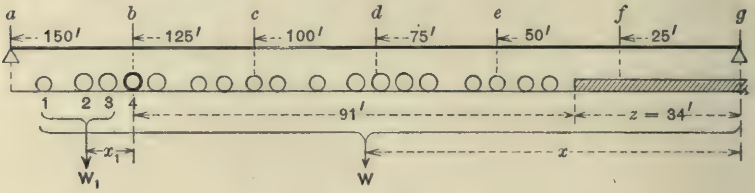


FIG. 35.

Example 1. Let it be required to find the position of Cooper's E–60 loading for maximum shear in the end panel of a 6-panel truss, 150 ft. long.

Solution. Referring to Fig. 35, apply the moment table with wheel 4 at b. The distance z covered by the uniform train load is then $125-91$ (from table) $=34$ ft.

$$W=426 \text{ (from table)} + (34\times3)=528,$$
$$W_1 \text{ (including wheel 4)}=105 \text{ (from table)},$$
$$W_1 \text{ (excluding wheel 4)}= 75 \text{ (from table)}.$$

Since $528 < 105\times6$ and $> 75\times6$, the criterion for maximum shear, Eq. (40), Art. 111, is satisfied. Or, dividing 528 by 6, and noting that the quotient must lie between 75 and 105, the criterion may be tested by a single operation.

For 3d wheel at b, $W=513$ and $W_1=45$ or 75.

For 5th wheel at b, $W=543$ and $W =105$ or 135.

For the former position one-sixth of W is greater than either value of W_1, and for the latter position one-sixth of W is less than either value of W_1, so that neither of these positions satisfies the criterion.

Example 2. Required the maximum floor-beam load from Cooper's E–60 Loading for panel lengths of 25 ft.

Solution. Lay off two panel lengths to the same scale as the moment table, as in Fig. 36. By Eq. (35), Art. 45, the criterion for maximum floor-beam load is $W=2W_1$, in which W is the total load on both panels, and W_1 is the load on panel AB.

Try wheel 13 at B, as shown in the figure. Wheel 10 is the first load to the right of A and wheel 17 is directly over C. This is the special case treated in Art. 112. When the loading is moved a differential distance to the right, wheel 17 passes to the right of C. The total load W is then the sum of loads 10 to 16, inclusive, which equals 174, and W_1 is the sum of loads 10 to 12, inclusive, or 75. When the loading is moved a differential distance to the left, wheel 17 passes to the left of C, and must now be included in W. Similarly wheel 13 must be included in W_1. Then $W=193.5$ and $W_1=105$. Since $174>2\times 75$ and $193.5<2\times 105$, the criterion is satisfied. The floor-beam load, by Eq. (39), Art. 45, is

$$174-\frac{720+794}{25}=113.4.$$

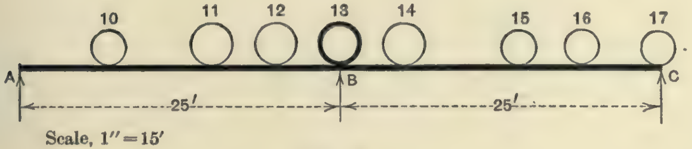


FIG. 36.

114. Application of Moment Table to Finding the Values of Moments. The moments Wx and W_1x_1 in Eqs. (38) and (41), Art. 111, may be conveniently found with the aid of the moment table, either by a simple calculation or by direct reading, in the manner explained in Art. 39. In general:

Let W = total live load on truss;
W_e = summation of engine loads on truss;
M = moment of W_e about head of uniform train load;
w = intensity of uniform train load;
z = length of uniform train load on bridge.

Then, referring to Fig. 35,

$$Wx=M+W_ez+\tfrac{1}{2}wz^2;$$

or

$$Wx=M+(W_e+\tfrac{1}{2}wz)z.$$

For Cooper's E–60 loading, $w=3$ in thousands of pounds per rail; hence for that loading,

$$Wx=M+(W_e+1.5z)z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

in which the values M and W_e are read directly from the moment table, and z is found as in the preceding article. If all wheel loads for both engines are on the bridge, as in Fig. 35, $W_e=426$

and $M = 24{,}550$, in thousands of pounds and thousands of foot-pounds, respectively, per rail. Again, for the position of loading shown in the figure $W_1x_1 = 720$. This value is also read directly from the table as the moment of wheels 10 to 12, inclusive, about 13, that being equal to the moment of wheels 1 to 3, inclusive, about 4, which is not given in the table.

115. Example. Pratt truss (Fig. 37), span length = 150 ft., height of truss = 28 ft; 6 panels. Required: Maximum live-load stresses for Cooper's E–60 Loading.

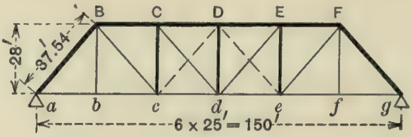


FIG. 37.

Solution. (*a*) *Diagonals.* It is desirable that operations similar in character should be performed in direct sequence. The positions of loading for maximum stress will first be found, noting in each case the distance, z, covered by the uniform train load, or, in the absence of the latter, the distance, z', from the last load on the bridge to the right end reaction.

Position of loading for maximum positive shear in successive panels. Lay off the span length and panel divisions, and find the positions of the loading which satisfy the criterion for maximum shear, $W = NW_1$, Eq. (40), Art. 111, with the aid of the moment table, as explained in Art. 113. The critical loads, and the corresponding values of W, W_1, z and z' are thus found to be as follows:

Panel.	Critical Wheel.		Last Wheel.	z	z'	W	W_1
	No.	at					
ab	4	*b*	. .	34	. .	528	75 or 105
bc	3	*c*	. .	4	. .	438	45 or 75
	4	*c*	. .	9	. .	453	75 or 105
cd	3	*d*	15	. .	0	348 or 367.5	45 or 75
de	2	*e*	10	. .	2	228	15 or 45
ef	2	*f*	6	. .	1	154.5	15 or 45

In each case, one-sixth of W is seen to be intermediate between the two corresponding values of W_1, as required by the criterion. For panel *bc*, the criterion is met when either wheel 3 or 4 is placed at the critical point *c*. The stresses for both positions will have to be computed, since it cannot be foreseen which value will prove greater. For panel *cd*, $z' = 0$, i.e., wheel 15 is at the right support, *g*, when the critical wheel, 3, is at *d*. This is the special case discussed in **Art. 112.** Since $6\times45<348$ and $6\times75>367.5$, the criterion is met.

With the aid of the moment table the shears, *V*, Eq. (38), Art. 111, may now be derived as follows, with due regard to Eq. (50) of the preceding article when the loading is partly uniform. The values are all expressed in thousands of pounds.

Panel	*R*	− *r*	= *Shear*
ab:	$[24{,}550+(426+\overline{1.5\times34})\times34]\div150=271.8$	720/25	243.0
bc:	$[24{,}550+(426+\overline{1.5\times4}\;)\times\;4]\div150=175.2$	345/25	161.4
	$[24{,}550+(426+\overline{1.5\times9}\;)\times\;9]\div150=190.1$	720/25	161.3
cd:	$16220\div150=108.1$	345/25	94.3
de:	$(6950+\overline{228\times2})\div150=49.4$	120/25	44.6
ef:	$(2460+\overline{154.5\times1})\div150=17.4$	120/25	12.6

The shears in panel *bc* for wheels 3 and 4, respectively, at the critical point *c* are seen to be practically equal.

From the dimensions of the truss, the stresses are found from the shears as follows:

$$
\begin{aligned}
aB &= 243.0\times\frac{37.54}{28.0}=(-)325.8;\\
Bc &= 161.4\times\;\text{“}\;=(+)216.4;\\
Cd &= 94.3\times\;\text{“}\;=(+)126.4;\\
De &= 44.6\times\;\text{“}\;=(+)\;59.8;\\
Fe &= 12.6\times\;\text{“}\;=(-)\;16.9.
\end{aligned}
$$

The live-load tensile stress, 59.8, in counter *De* should be diminished by one-half the dead-load compressive stress in that member, assuming the main diagonal *Ed* in that panel to be omitted (Eq. (33), Art. 102).

The compressive counter stress, 16.9, in *Fe* is of interest only when the specifications require the use of minimum stresses for proportioning the truss members (Art. 101). The minimum stress in *Bc* and *Fe* is found by deducting 16.9 from the dead-load stress. The minimum stress in *Cd* and *De* is zero.

(*b*) *Verticals.* The stresses in the vertical posts are equal to the shears in the appropriate panels. Thus

$$
\begin{aligned}
Cc &= -94.3;\\
Dd &= -44.6.
\end{aligned}
$$

The stress in the hip vertical *Bb* is equal to the maximum floor-beam load at *b*. Its value as found in Art. 113, Example 2, is 113.4, whence

$$Bb=+113.4.$$

The minimum stresses in the vertical posts may be lower than the dead-load stresses, and may be found, if desired, as explained in the example, Art. 103.

(*c*) *Chords.* The stress in the end panels of the lower chord is equal to the horizontal component of stress aB, or it is found from the shear in the end panel and the dimensions of the truss, as follows:

$$ab(=bc)=243.0\times\frac{25}{28}=(+)217.0.$$

The positions of loading for maximum stresses $cd(=BC)$ and CD are found by the criterion, Eq. (43), Art. 111, with the aid of the moment table.

For cd, placing wheel 7 at c, $z=100-72=28$, $W=426+(28\times3)=510$, $W_1=154.5$ or 174. Since $510\times\frac{2}{6}$ is greater than 154.5 and less than 174, the criterion is satisfied.

For CD, placing wheel 12 at d, $z=75-40=35$, $W=426+(35\times3)=531$, $W_1=258$ or 288. Since $531\times\frac{3}{6}$ is greater than 258 and less than 288, the criterion is satisfied.

With the aid of the moment table the values of the bending moments, M, may now be found by Eq. (41), Art. 111 (expressing a and l in panel lengths), and the stresses follow by dividing these bending moments by the height of the truss. Thus

$$\begin{array}{lrr}
 & M \div 28 = & \textit{stress} \\
cd=BC:\ [24{,}550+(426+\overline{1.5\times28})\times28]\frac{2}{6}-\ 3{,}230= & 9{,}321 & (\pm)332.9 \\
CD:\ [24{,}550+(426+\overline{1.5\times35})\times35]\frac{3}{6}-10{,}060= & 10{,}589 & (-)378.2.
\end{array}$$

116. Approximate Check for Live-Load Stresses.

It is important to guard against gross numerical errors by applying approximate independent checks to the computed stresses.

(*a*) *Web Stresses.* In Art. 98, it was seen that for live-load web stresses from uniform loading obtained by the conventional method the second differences of the stresses are constant. It will be shown later (Art. 124) that this condition is met also by the web stresses from uniform loading computed by the exact method, and it is fulfilled approximately for web stresses based on a concentrated load system. Thus, considering the values obtained in the example in the previous article, rounded off to the nearest thousand pounds:

Stresses.	1st Differences.	2nd Differences.
326		
	110	
216		20
	90	
126		24
	66	
60		23
	43	
17		

If stress 126 is increased by 5, the second differences will be 25, 14, 28, and if stress 126 is decreased by 5, the second differences will be 15, 34, 18. Important errors, especially in the intermediate stresses, are thus readily detected. This check is applicable only to the web stresses in trusses with horizontal chords. Another method of checking web stresses will be given later (Art. 127).

(*b*) *Chord Stresses.* The live-load chord stresses, S_l, may be checked approximately by multiplying each dead-load stress, S_d, by the ratio S_l/S_d for the end chord member. The resulting stresses should not exceed the maximum computed values of S_l by more than 5 or 6 per cent. This check is applicable to chord stresses of simple trusses in general. For trusses with horizontal chords, the coefficients for the dead-load stresses may be used instead of the stresses themselves. Applying this check to the live-load stresses found in the preceding article,

	Stresses. Approx.	Stresses. Actual.
$cd(=BC)=4.0\times\frac{217.0}{2.5}=$	347.2	332.9
$CD=4.5\times\frac{217.0}{2.5}=$	390.6	378.2

The percentage of error, on the basis of the actual stresses, is seen to be about 4 for *cd* and 3 for *CD*.

117. Critical Wheel for Maximum Web Stress. The criterion for maximum shear in any panel of a truss of N equal panels, as derived in Art. 111, Eq. (40) is

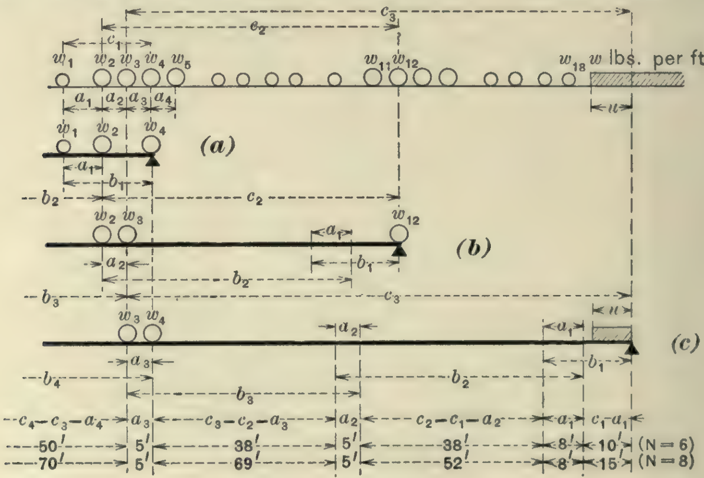
$$W=NW_1,$$

in which W denotes the total live load on the truss and W_1 the total load in the panel. As stated in that article, the critical wheel is always a driver of the first engine, but sometimes two successive drivers will satisfy the criterion. One such instance occurred in the example in Art. 115. The conditions under which any one wheel or two successive wheels will meet the criterion will now be considered.

In Fig. 38, w_1, w_2, w_3, etc., denote successive wheels from left to right. If these wheels are assumed to act in turn, first a differential distance to the right and then to the left of the critical point, each wheel will meet the criterion, or 'govern' under the following conditions:

Critical Wheel.	For critical wheel to the *right* of critical point, W must be *greater* than:	For critical wheel to the *left* of critical point, W must be *less* than:
w_1	zero	Nw_1
w_2	Nw_1	$N(w_1+w_2)$
w_3	$N(w_1+w_2)$	$N(w_1+w_2+w_3)$
. . .	. . . .	. . . .
w_n	$N(w_1+w_2+\ldots+w_{n-1})$	$N(w_1+w_2+\ldots+w_n)$

It is seen therefore that any wheel in the first column of the table will cease to govern as soon as the total load on the truss exceeds the value in the last column. Let the sum of any series of successive loads, from w_1 to w_4 (inclusive), for example, be designated $\Sigma(w_1 \text{ to } w_4)$, and let w denote the intensity of the uniform train load per truss. Suppose that the following relations exist for a given value of N (as they do for $N=6$ and Cooper's E–60 Loading):



c_3
c_2
c_1
w_1 w_2 w_3 w_4 w_5
w_{11} w_{12}
w_{18} w lbs. per ft.
a_1 a_2 a_3 a_4
u
(a)
b_1
b_2
(b)
b_3
(c)
b_4
$c_4-c_3-a_4$ a_3 $c_3-c_2-a_3$ a_2 $c_2-c_1-a_2$ a_1 c_1-a_1
50′ 5′ 38′ 5′ 38′ 8′ 10′ (N = 6)
70′ 5′ 69′ 5′ 52′ 8′ 15′ (N = 8)

Fig. 38.

$$[W=\Sigma(w_1 \text{ to } w_3)] < Nw_1 \quad \text{and} \quad [W=\Sigma(w_1 \text{ to } w_4)] > Nw_1;$$

$$[W=\Sigma(w_1 \text{ to } w_{11})] < N(w_1+w_2) \text{ and } [W=\Sigma(w_1 \text{ to } w_{12})] > N(w_1+w_2);$$

$$[W=\Sigma(w_1 \text{ to } w_{18}) + uw] = N(w_1+w_2+w_3).$$

Wheels w_1, w_2 and w_3 will then cease to govern as soon as the loading advances further to the left from the positions indicated in Figs. (*a*), (*b*) and (*c*) respectively. Thus wheel w_1 governs only for the distance b_1, Fig. (*a*)—that is to say, w_1 will govern at any point to the right of the position it occupies in that figure, whereas w_2 will govern at points along b_2 to its left. Hence w_1 and w_2 will both govern at any point of the intervening distance a_1, as may be seen by assuming the loading shifted so as to place in turn each of these two wheels at such a point, and noting that the conditions demanded by the criterion and stated in the above table are satisfied in each case. The limiting distance to the left, within which w_2 governs, is shown in Fig. (*b*), in which w_{12} is at the right support. In that figure w_2 and w_3 will govern to the right and left, respectively, of the positions in which they are shown, so that both will govern over the distance a_2. The relation of Fig. (*c*) to Fig. (*b*) is similar to that of Fig. (*b*) to Fig. (*a*). In this way the distances b_1, b_2, b_3, etc., dominated respectively by w_1, w_2, w_3, etc., may be determined for any value of N, each distance overlapping the next, owing to the fact that as one wheel ceases to be the critical wheel, the next wheel back becomes the critical wheel. The distances which each wheel dominates exclusively may be expressed in terms of $c_1(=b_1)$, c_2, c_3, etc., and a_1, a_2, a_3, etc., in the manner shown in Fig. (*c*). The corresponding numeric values for Cooper's E–60 loading are also given for trusses of 6 and of 8 equal panels. The latter number is rarely exceeded in railway bridges with horizontal chords. From these figures it is seen that the third and fourth wheels (the two middle drivers) usually govern for maximum shears—i.e., positive shears in the left half of the truss—and the second wheel (first driver) for counter shears—i.e., positive shears in the right half of the truss—as in the example of a six-panel truss in Art. 115. In the latter, the third and fourth wheels were both found to govern for maximum shear in panel *bc*, the critical wheel being at *c*, 100 ft. from the right end support. In Fig. 38 this point, for $N=6$, will be found to lie in the 5-ft. space a_3, in which w_3 and w_4 both govern.

From the foregoing it is seen that the criterion for maximum shears is as a rule fulfilled by only a single wheel; but that if the critical point happens to lie in one of the short stretches governed by two successive wheels, the stress for both positions should, strictly speaking, be computed and the greater value used. However, the difference between the two stresses is relatively so small that in practice usually no attempt is made to ascertain whether a second position will satisfy the criterion for maximum web stresses.

118. Position of Loading for Maximum Chord Stress. As stated in Art. 111, several positions of loading which satisfy the criterion for maximum chord stress can usually be found. A little practice and judgment are required to select the position which will give either the greatest stress or a stress differing from the same so slightly that the difference is of no practical importance. In practice the stress in a given chord member is seldom computed for more than one position of loading.

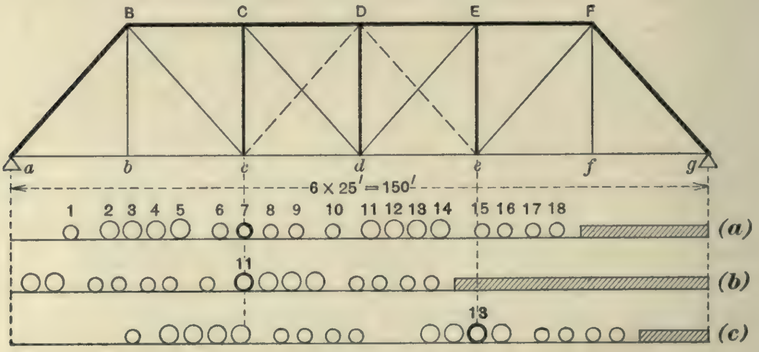


FIG. 39.

Fig. 39 represents the truss for which the live-load stresses were obtained in Art. 115. For the load advancing from the right, wheel 4 is the first load which, on reaching the critical point *b*, satisfies the criterion for maximum stress *ab*(=*bc*). If the load is advanced further, the criterion is next satisfied when wheel 10 reaches *b*, and it is also satisfied for wheel 11, 12 or 13 at *b*. It may be seen in advance, however, that the governing position is that for which wheel 4 is the critical load, since for the other positions the heavy drivers 2, 3, 4 and 5, are off the bridge.

Again, for chord stress *cd*=*BC* the critical point is at *c*, and wheel 7 is the first load which will satisfy the criterion. The corresponding position of the loading is shown to scale in Fig. (*a*). It is seen that for that position the heavy drivers are not in the immediate vicinity of the moment center, whereas, generally speaking, the bending moment is greatest when that condition is also met. If the load is advanced to position (*b*), with the first driver, 11, of the second engine at *c*, the drivers of the first engine are either off the span, or so near the left support as to have little influence on the bending moment at *c*. If, however, the load is placed as in Fig. (*c*) with wheel 13 at the critical point *e* for stress $de = EF$ corresponding to $cd = BC$ in the other half of the truss, the criterion will be satisfied and the resulting stress is apt to be somewhat greater than for position (*b*), since one set of drivers is near the moment center and the other set is so situated as to produce considerable bending moment at *e*. For practical purposes it would be sufficient to compute the stress for position (*c*) alone, since it is either the greatest stress or only slightly less than the greatest.

The various positions which satisfy the criterion and the corresponding stresses in thousand pounds are shown in the following table. In the light of the foregoing the choice of positions for maximum would be limited in advance to those for which the critical loads are printed in heavier type.

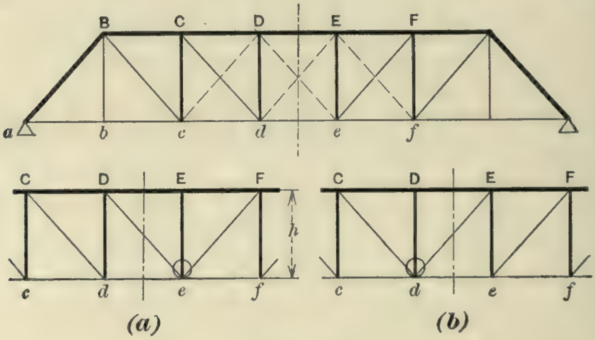
LIVE-LOAD CHORD STRESSES

$ab = bc$ or $fg = ef$				$BC = cd$ or $EF = de$				$CD = DE$			
Critical		Stress.	Ratio.	Critical		Stress.	Ratio.	Critical		Stress.	Ratio.
Wheel	Pt.			Wheel	Pt.			Wheel	Pt.		
4	*b*	217.0	1.000	**13**	*e*	334.0	1.000	**12**	*d*	378.2	1.000
12	*b*	199.8	0.921	**14**	*e*	333.8	0.999	**11**	*d*	377.5	0.998
11	*b*	199.4	0.919	**7**	*c*	332.9	0.997	..	..	.....	.....
13	*b*	198.3	0.914	8	*c*	332.5	0.995	..	..	.....	.....
10	*b*	196.1	0.904	11	*c*	321.9	0.964	..	..	.....	.....
..	..	.....	.....	12	*c*	321.0	0.961	..	..	.....	.....
..	..	.....	.....	13	*c*	319.6	0.957	..	..	.....	.....

Stress *cd* for wheel 7 at *c*, which has the value 332.9, as found in the example in Art. 115, is seen to be slightly less than stress *de* for wheel 13 at *e*, which equals 334.0 and corresponds to a like stress in *cd* for the loading reversed and advancing from left to right.

Problem. Prove with the aid of the moment table that the criterion for maximum chord stress is satisfied when the loading occupies any of the positions indicated in the above table of live-load chord stresses.

119. Special Case. Chord Stresses in Middle Panel of Truss with Odd Number of Panels. When the number of panels is odd, the analysis of the chord stresses in the middle panel involves certain special features. Referring to Fig. 40, and assuming the counters to be flexible members capable of acting in tension only, then when the criterion for maximum bending moment is fulfilled with the critical load at *d* or *e*, the shear, *V*, in the middle panel may be either positive or negative, according to circumstances which cannot be foreseen. When *V* is positive counter *De*, Fig. (*a*), acts, and when the shear is negative, counter *Ed*, Fig. (*b*), acts. When the live load occupies the position for maximum stress in any chord member, the shears in all panels to the left of the center panel are positive, and in all panels to the right negative, so that counters *Dc* and *Ef* are not brought into action. These members are therefore omitted in Figs. (*a*) and (*b*).



(*a*) (*b*)

FIG. 40.

Letting S_h denote the horizontal component of the stress in counter *De*, and applying $\Sigma H = 0$ to the forces at *D*, Fig. (*a*):

$$\text{Stress } DE(=EF) = \text{stress } CD(=de) + S_h.$$

Similarly, if S_h denotes the horizontal component of the stress in counter *Ed*, Fig. (*b*):

$$\text{Stress } DE(=CD) = \text{stress } EF(=de) + S_h.$$

It is thus seen that for the load in any position, except one causing the shear in the middle panel to become zero, the stress in the middle panel and one adjacent panel of the upper chord exceeds the stress in the middle panel of the lower chord by the horizontal component of the stress in one of the middle counters.

For maximum stress in any chord member of a Pratt truss (or, as will appear later, in any truss with vertical web members) the critical weight must be placed at the vertical which passes through the moment center of the member in question, as was seen in Art. 111 (c), and the criterion, Eq. (42), must be satisfied.

Upper Chord Stress. Letting M_d and M_e denote the bending moments at d and e, respectively, the requirements for maximum stress in DE are met under either of the following conditions:

First. When with the critical wheel at e, as in Fig. (*a*), (1) the criterion for maximum M_e is satisfied; and (2) V is *positive*, so that counter De acts and the moment center lies at e, as was assumed in advance in selecting e as the critical point. Stress DE is then M_e/h.

Second. When with the critical wheel at d, as in Fig. (*b*), (1) the criterion for maximum M_d is satisfied; and (2) V is *negative*, so that counter Ed acts and the moment center lies at d, as was assumed in advance in selecting d as the critical point. Stress DE is then M_d/h.

Suppose that with the loading headed toward the left, M_d and M_e are computed for each position which satisfies the criterion for maximum, and suppose that the greatest possible bending moment is found to occur when a certain critical load is at e. For the loading in that position $M_e > M_d$ and V is necessarily *positive* (Art. 33). Suppose, on the other hand, that the greatest possible bending moment is found to occur when a certain critical load is at d. For the loading in that position $M_d > M_e$ and V is necessarily *negative*. In either case, therefore, requirement (2) is met. Therefore, in general, if that position of loading is found for which M_d or M_e attains the greatest possible value, requirement (2) is necessarily satisfied, and V need not be computed with a view of ascertaining whether it is positive or negative.

When stress DE is maximum, the same maximum stress is developed simultaneously in one of the chord members adjacent to DE—i.e., EF when counter De is in action, Fig. (*a*); or CD

when counter *Ed* is an action, Fig. (*b*). When the loading is reversed, and similarly disposed with respect to the center line of the truss, the stress in the other chord member adjacent to *DE* becomes maximum. The maximum stresses in *CD*, *DE* and *EF* are thus seen to be equal.

Lower Chord Stress. The conditions for maximum stress in *de* follow by analogy with the foregoing.

First. When with the critical wheel at *d*, Fig. (*a*), (1) the criterion for maximum M_D is satisfied; and (2) V is *positive*, so that counter *De* acts and the moment center lies at *D*, as was assumed in advance in selecting the critical point on the vertical through *D*. Stress *de* is then M_D/h.

Second. When with the critical wheel at *e*, Fig. (*b*), (1) the criterion for maximum M_E is satisfied; and (2) V is *negative*, so that counter *Ed* acts and the moment center lies at *E*, as was assumed in advance in selecting the critical point on the vertical through *E*. Stress *de* is then M_E/h.

For practical purposes the maximum stress in *de* may be assumed equal to that found for the upper chord. This assumption involves usually a slight error on the side of safety.

Loading for zero shear in middle panel. The upper chord stresses *CD*, *DE*, and *EF*, and the lower chord stress *de* are equal for any position of loading which causes the shear in the middle panel to be zero. For that loading the bending moment is constant throughout the middle panel, and neither of the counters is in action. Such a position of loading may readily be found by trial, as will be shown in the following example. The resulting lower chord stress in the middle panel is sometimes greater than that for the conditions of loading previously discussed.

Example. Find the maximum chord stresses in the middle panel of the truss shown in Fig. 41 for Cooper's E–60 loading.

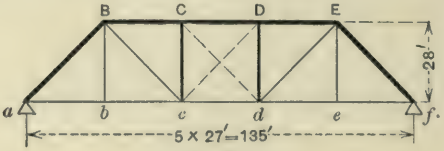


FIG. 41.

Solution. Several positions of the loading which satisfy the above conditions (1) and (2) for maximum upper and lower chord stress are indicated in the following table. The number and position of the critical wheel, the character and magnitude of the shear in the middle panel, and the resulting chord stress are also shown:

Critical Wheel.		Shear in Panel *cd*	Acting Counter.	Chord Member.	Stress in Thousands of lbs.
No.	at				
8	*c*	+ 6.8	*Cd*	*cd*	+294.0
9	*c*	+ 5.8	*Cd*	*cd*	+293.6
10	*c*	− 0.4	*Dc*	*CD*	−289.6
12	*c*	−13.0	*Dc*	*CD*	−287.4
12	*d*	+ 7.5	*Cd*	*CD*	−300.7
13	*d*	+ 6.9	*Cd*	*CD*	−300.3

With the critical wheel at *c*, two positions give positive shear in the middle panel and two positions negative shear. With the critical wheel at *d*, both positions produce positive shear in the middle panel, and no position which satisfies the criterion gives negative shear in that panel. The maximum stress in the upper chord is seen to be −300.7, and in the lower chord +294.0. If the former value is adopted also for the lower chord, as is customary in practice, the error is seen to be about 2 per cent on the side of safety.

The greatest possible bending moment is developed when wheel 12 is at *d*, i.e., at the right end of the middle panel. For that position the shear in the middle panel is necessarily positive, as previously explained.

Lower chord stress *cd* will also be found for the loading in a position for zero shear in the middle panel. For wheel 9 at *c* the shear in that panel, as given in the table, is +5.8, and for wheel 10 at *c* the shear is −0.4. Hence for zero shear wheels 9 and 10 must be on opposite sides of *c* and wheel 10 must be nearly at *c*. If wheel 10 is 1 ft. to the right of *c* (trial position) the shear in panel *cd* is +0.9, i.e., 1.3 greater than −0.4, the shear for wheel 10 at *c*. For zero shear, wheel 10 should therefore be placed 0.4/1.3, or 0.3 ft. to the right of *c*. The corresponding stress *cd*, found by moments about *C* or *D*, is +290.4, i.e., less than the value +294.0 found by the first method.

120. Howe Truss. The analysis of the maximum live-load stresses for the Howe truss, Fig. 28, Art. 106, is precisely the same as for the Pratt truss, Art. 111, the resulting stresses in the diagonals being compressive instead of tensile. For the stresses in the verticals the general relations exist to which attention was called in Art. 106.

121. Triangular Truss. (*a*) *Web Stresses.* The analysis of the maximum live-load web stresses for the triangular truss, with or without verticals, Fig. 29, Art. 107, is the same as for the Pratt truss, Art. 111 (*a*).

(*b*) *Chord Stresses.* The analysis of the maximum live-load chord stress for the triangular truss with verticals supporting the loaded chord is the same as for the Pratt truss, Art. 111 (*c*), except that for the former, the critical points for the upper and lower chord stresses do not lie in the same vertical plane, so that no equality exists between these stresses.

For a triangular truss without verticals, the criterion for maximum chord stress for the *loaded chord* differs from that for the Pratt truss.

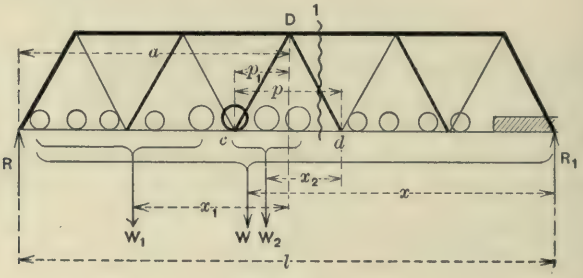


FIG. 42.

In Fig. 42, consider any member of the loaded chord, as *cd*, for example. Taking a section, 1, the moment, M, of the forces to the left of the section about the moment center D has the value

$$M=\frac{Wx}{l}a-W_1x_1-\frac{W_2x_2}{p}p_1. \quad \ldots \quad (51)$$

For $M_{\max}$,

$$\frac{dM}{dx}=\frac{Wa}{l}-W_1-\frac{W_2}{p}p_1=0,$$

or

$$W=\frac{l}{a}\left(W_1+\frac{p_1}{p}W_2\right). \quad \ldots \quad (52)$$

For the usual case of equal panels, and diagonals of equal inclination, n denoting the number of panels in a, and N the total number in the truss:

$$W=\frac{N}{n}\left(W_1+\tfrac{1}{2}W_2\right). \quad \ldots \quad (53)$$

The critical load may be placed at either c or d, since a differential movement from either position will affect the value of the second member of Eq. (53).

After the position of loading which satisfies the criterion has been ascertained, M is obtained by Eq. (51). The moments Wx, W_1x_1 and W_2x_2 may be found with the aid of the moment table. The chord stress is then M/h.

For the *unloaded chord*, the moment centers lie at the joints of the loaded chord. The expression for M is therefore precisely the same as for the Pratt truss, Eq. (41), Art. 111; hence the criterion for maximum stress in the unloaded chord is the same as for the Pratt truss.

122. Trusses with More than One Web System. In trusses with more than one web system, such as the Whipple truss, and the double and quadruple triangular trusses, the ambiguity of stress due to the presence of one or more redundant members is such that refinements in seeking the position for maximum live-load stress in any member are wholly unwarranted.

(*a*) *Web Stresses.* The maximum web stresses from a concentrated load system may be found with sufficient accuracy by considering one of the middle drivers of the first engine placed at a floor beam and computing the reactions, r_1, r_2, r_3, etc., at that and successive floor beams for the loading in that position. These reactions are readily found with the aid of the moment table by the method explained in Art. 46. The original load system may then be regarded as having been transformed into an 'equivalent' load system consisting of a series of loads (the successive floor-beam reactions) separated by intervals of one panel length. This new load system may then be applied to the component trusses, the loads r_1, r_2, r_3, etc., being placed at suitable panel points for maximum shear in each panel in turn. Only those loads should be used which correspond to the joints of the component truss under consideration. Thus in Fig. 43, in which the two web systems are distinguished by the solid and broken lines, for maximum stress $Bc(=Dc)$, loads r_1, r_3 and r_5 are assumed to act at D, F and H. For maximum counter stress in these members, the stress in the corresponding members Fg and Hg in the other half of the truss, for a single load r_1 at H, is computed.

It is desirable that the panel length of the component trusses should not be an even fractional part of the distance between the centers of gravity of the drivers of the first and second engines (56 ft. for Cooper's load system), since otherwise the heavy panel concentrations from the drivers of both engines would act on each component truss.

(*b*) *Chord Stresses.* The chord stresses may be found with sufficient accuracy for practical purposes on the basis of conventional load systems consisting of a so-called 'equivalent uniform load,' or of a uniform load with one or two 'excess loads,' which will give substantially the same bending moments throughout the truss as the concentrated load system. The chord stresses for a uniform live load covering the entire bridge may be found as explained in connection with the analysis of dead-load stresses of these types. If one or two excess loads are used in addition to the uniform load, the points at which these excess loads should be assumed to act for maximum stress may readily be found by inspection, as will be seen later in the treatment of conventional load systems (Chap. XII).

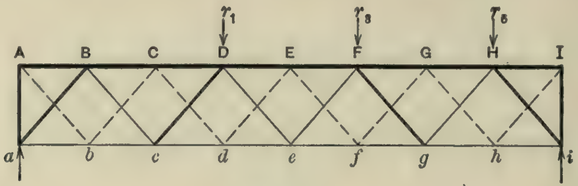


FIG. 43.

The live-load web stresses may also be obtained by the use of conventional load systems, but the agreement between the resulting stresses and those found by the method above described is, generally speaking, less satisfactory than for chord stresses.

E. STRESSES FROM UNIFORM LIVE LOAD. EXACT METHOD

123. Introduction. The analysis of live-load web stresses from uniform loading by the conventional method has been considered in Section C of this chapter. The errors by this method are always on the side of safety. While the conventional method is frequently used on account of its supposed convenience, it will be seen that for trusses with horizontal chords and single web systems, the web stresses may be obtained just as readily by the exact method which will now be considered.

Since the chord stresses are all maximum when the uniform load covers the entire bridge, these stresses require no further attention.

124. Pratt Truss. It was shown in Art. 98 that the positive shear in any panel of a truss (Fig. 44), as *cd*, for example, becomes maximum when the uniform load extends from the right support a distance x into the panel given by the equation

$$x=\frac{ap}{l-p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

The criterion for maximum shear from concentrated loads, Eq. (39), Art. 111, is perfectly general, and a uniform load may be regarded as a special case of concentrated loading (Art. 38). Hence in the present case also, for maximum shear:

$$W=\frac{l}{p}W_1,$$

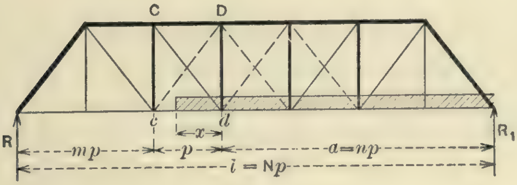


FIG. 44.

in which W is the total load on the bridge and W_1 the total load in the panel in question. Assuming unity as the intensity of the uniform load in Fig. 44, $W=a+x$ and $W_1=x$, whence

$$a+x=\frac{l}{p}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

and

$$x=\frac{ap}{l-p},$$

which agrees with Eq. (54), as it should.

The shear, V, in panel *cd*, for a uniform load of intensity unity, has the value

$$V=\frac{(a+x)^2}{2l}-\frac{x^2}{2p}.$$

Substituting for $(a+x)$ from Eq. (55),

$$V=\frac{lx^2}{2p^2}-\frac{x^2}{2p},$$

or

$$V=\frac{l-p}{2p^2}\,x^2.$$

Substituting for x from Eq. (54)

$$V=\frac{a^2}{2(l-p)},$$

or, in general, for a uniform load of intensity w,

$$V=\frac{a^2w}{2(l-p)}. \qquad (56)$$

For the usual case of equal panels, letting $a=np$ and $l=Np$, Eq. (56) becomes

$$V=\frac{wp}{2(N-1)}\,n^2. \qquad (57)$$

For any given truss and loading, the factor $wp/2(N-1)$ is a constant, so that the maximum shear, V, may readily be evaluated for successive panels by a single setting of the slide rule.

Expressing a and l in terms of p in Eq. (54),

$$x=\frac{p}{N-1}\,n.$$

The load in the panel is then

$$wx=\frac{wp}{N-1}\,n. \qquad (58)$$

From Eqs. (57) and (58) $V=wx\times\frac{1}{2}n.$ Hence, when the criterion for maximum positive shear in any panel is satisfied, the shear itself is the product of the load in the panel by one-half the number of whole panels to the right.

Comparison of Formulas for Exact and Conventional Methods. The maximum shears from uniform loading by the conventional method for a truss of equal panels are expressed by Eq. (17), Art. 98, as follows:

$$V=\frac{wp}{2N}\,n(n+1). \qquad (59)$$

Comparing this formula for maximum shears with that by the exact method, Eq. (57), for the right end panel, $n=$ zero and V by both formulas is zero. For the left end panel, $n=N-1$ and V by both formulas equals $wp(N-1)/2$. This is the value of the left end reaction when the uniform load extends over the whole bridge, which is obviously the condition for maximum positive shear in the left end panel. For all intermediate panels the errors by the conventional method are on the side of safety. A general expression for the relative error, e, may be found by subtracting the value of V by Eq. (57) from that by Eq. (59) and dividing the difference by the former, whence

$$e=\left[\frac{n(n+1)}{N}-\frac{n^2}{N-1}\right]\frac{N-1}{n^2}.$$

Observing from Fig. 44, that $m=N-n-1$, and reducing,

$$e=\frac{m}{Nn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

For the left end panel $m=$zero, therefore $e=$zero. For intermediate panels e varies with m/n and since for successive panels to the right m increases and n decreases, e increases rapidly from left to right. Since the positive shear in the right end panel is zero, e is maximum in the second panel from the right, for which $m=N-2$ and $n=1$, whence $e=(N-2)/N$. If $N=7$, $e=\frac{5}{7}$. In general, since e for this panel equals $(N-2)/N$ the maximum error approaches unity, or 100 per cent, for increasing values of N. For the middle panel of a seven-panel truss $m=n=3$, and $e=\frac{1}{7}$. While the errors increase rapidly in successive panels toward the right the shears themselves decrease rapidly in that direction, so that the errors, considered absolutely and not relatively, are not serious.

The absolute error, q, or the difference in the values of V by Eqs. (57) and (59), is

$$q=\frac{wp}{2N(N-1)}nm, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and since $n+m=N-1$ (a constant), q is maximum for $n=m$ $=(N-1)/2$. Therefore

$$q_{\max}=\frac{wp}{8}\,\frac{N-1}{N}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Eq. (62) shows that the maximum absolute error in the shears computed by the conventional method, which occurs in the middle panel, is always somewhat less than $\frac{1}{8}wp$, and that for w and p constant, its exact value is only slightly affected by N, and therefore, by Np, the span length.

The maximum live-load shears from uniform loading by the conventional method and by the exact method may be conveniently tabulated for a panel load of unity ($wp=1$) by evaluating the factor $n(n+1)/2N$ in Eq. (59), and the factor $n^2/2(N-1)$ in Eq. (57). In the following table the upper values denote the factors by the conventional method and the lower values by the exact method. The maximum stresses in the diagonals are the products of these coefficients and wph'/h, in which h' is the length of the diagonal and h the height of the truss.

COEFFICIENTS FOR MAXIMUM LIVE-LOAD SHEARS FROM UNIFORM LOADING

N	Panel										
	ab	*bc*	*cd*	*de*	*ef*	*fg*	*gh*	*hi*	*ik*	*kl*	*lm*
4	1.50 1.50	0.75 0.67	0.25 0.17								
5	2.00 2.00	1.20 1.13	0.60 0.50	0.20 0.13							
6	2.50 2.50	1.67 1.60	1.00 0.90	0.50 0.40	0.17 0.10						
7	3.00 3.00	2.14 2.08	1.43 1.33	0.86 0.75	0.43 0.33	0.14 0.08					
8	3.50 3.50	2.63 2.57	1.88 1.79	1.25 1.14	0.75 0.64	0.38 0.29	0.13 0.07				
9	4.00 4.00	3.11 3.06	2.33 2.25	1.67 1.56	1.11 1.00	0.67 0.56	0.33 0.25	0.11 0.06			
10	4.50 4.50	3.60 3.56	2.80 2.72	2.10 2.00	1.50 1.39	1.00 0.89	0.60 0.50	0.30 0.22	0.10 0.06		
11	5.00 5.00	4.09 4.05	3.27 3.20	2.55 2.45	1.91 1.80	1.36 1.25	0.91 0.80	0.55 0.45	0.27 0.20	0.09 0.05	
12	5.50 5.50	4.58 4.55	3.75 3.68	3.00 2.91	2.33 2.23	1.75 1.64	1.25 1.14	0.83 0.73	0.50 0.41	0.25 0.18	0.08 0.05

Example. In the example in Art. 103 the live-load web stresses in a highway bridge were computed by the conventional method from the following data: Span length $=128$ ft., panel length $=16$ ft., height of truss $=20$ ft., live load per foot of bridge $=1500$ lbs.

The maximum live-load shears in this truss will now be computed by the exact method and compared with the shears previously obtained by the conventional method.

Solution. Let the panel loads and shears be expressed, as before, in thousands of pounds.

Panel load, $wp=\frac{1}{2}(1.5\times16)=12$.

The constant factor in Eq. (57) has the value

$$\frac{wp}{2(N-1)}=\frac{12}{2\times7}=0.857.$$

In the following table the maximum positive shears by the exact method are given in column (a), and the values previously found by the conventional method in column (b).

MAXIMUM POSITIVE SHEARS

Panel	(a)	(b)	Differences $(b)-(a)$	Differences Per cent
ab,	$0.857\times7^2=42$	42	0.0	0
bc,	" $\times6^2=30.9$	31.5	0.6	2
cd,	" $\times5^2=21.4$	22.5	1.1	5
de,	" $\times4^2=13.7$	15	1.3	9
ef,	" $\times3^2=7.7$	9	1.3	17
fg,	" $\times2^2=3.4$	4.5	1.1	32
gh,	" $\times1^2=0.86$	1.5	0.64	74

By Eq. (62) the maximum error in the two middle panels should be $\frac{12}{8}\times\frac{7}{8}=1.3$, which agrees with the values for these panels in the table. It is also seen that while the percentage of error increases rapidly in the panels from left to right, the absolute errors are inconsiderable and decrease from the middle panels toward both ends, as was to be anticipated.

It is seen that the computation of the maximum shears by the exact method is just as simple as by the conventional method. The stresses in the diagonals, instead of shears, might have been found by a single multiplication for each member if the constant 0.857 had first been multiplied by 25.61/20, the secant of the angle between the diagonal and a vertical. In a truss in which the principal web members are partly verticals it is desirable, however, to derive the stresses in the diagonals from the shears by a separate computation, since the shears are then the stresses in the verticals.

The variable factor, n^2, in Eq. (57) denotes the square of successive integers having the values (beginning at the left end panel) $N-1$, $N-2$,

$N-3$, etc. Letting the constant $wp/2(N-1)$ equal c, the second differences of the values cn^2 are constant and equal to $2c$. Where the series of values exceeds 3 in number, the correctness of the results may therefore be readily checked by ascertaining whether their second differences are constant as in the case of values found by the conventional method (Art. 98). Thus the second differences of the values in column (a) in the above table of shears should be $0.857\times2=1.7$(about). Their actual range is from 1.6 to 1.8.

The combined conditions of parallel chords, single web system, and uniform live loading required for this method of computing web stresses limit its applicability, in so far as main trusses are concerned, to highway bridges of moderate span lengths. The conditions mentioned are usually fulfilled, however, by the lateral truss system in the plane of the loaded chords for bridges in general, the uniform load being that due to the wind pressure assumed to be exerted horizontally on the moving load.

125. Triangular Truss. The general formula for maximum shear in the Pratt truss from uniform live loading, Eq. (57), of the preceding article is equally applicable to the triangular truss, Fig. 7, Art. 87, the value of p being equal to the distance between two successive joints in the loaded chord, designated by $2p$ in the figure. If the loaded chord is supported by verticals, Fig. 8, Art. 88, p equals the horizontally projected length of a single diagonal. Since the stresses in the verticals are not equal to the shears there is, in this case, no need of computing the latter. Letting S denote the maximum stress in any diagonal,

$$S=\frac{wph'}{2(N-1)h}n^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

in which n is the only variable for successive panels in a given truss.

126. Comparison of Web Stresses from Uniform and Concentrated Loading. In concentrated load systems the average weight per foot of the engines and tenders is usually considerably greater than that of the uniform train load. Thus for Cooper's E–60 loading the value of the former per rail in thousands of pounds is $426\div109=3.91$, whereas the train weighs 3.0 per foot, per rail. In general, the average live load per foot of the distance covered by the loading becomes greater as that distance decreases. Hence a uniform load which, advancing from the right, gives the same maximum shear in the left end panel of a truss as a given concentrated load system, will give smaller maximum shears than the latter in all other panels, the differences becoming relatively greater as the loaded length decreases. This is always true if the shears from such a uniform load are computed by the exact method. These differences are, however, more than neutralized by the errors on the side of safety involved in the conventional method of treating uniform loads (Art. 124). That is to say, the maximum web stresses from a concentrated load system are always intermediate between those derived from a so-called ‘equivalent’ uniform load by the exact and by the conventional method, the equivalent uniform load having such a value that it gives the same maximum shear in the end panel as the concentrated load system.

127. Approximate Check for Live-Load Web Stresses from a Concentrated Load System. A method by which the maximum stresses in the diagonals of a truss with horizontal chords, from a concentrated load system, may be approximately checked was given in Art. 116. The relations between these stresses and those from an ‘equivalent’ uniform load, referred to in the preceding article, afford another means of checking such web stresses with the aid of the table of coefficients in Art. 124. The use of such a table in that connection was first pointed out by C. W. Hudson.

From the example in Art. 115 the maximum stresses from Cooper's E–60 loading, in the successive diagonals of a six-panel, 150-ft. truss are 325.8, 216.4, 126.4, 59.8 and 16.9. If the stress, 325.8, in the inclined end post had been produced by a uniform load, then $325.8 \div wph'/h$ would equal 2.5, the coefficient in the table for panel *ab* in a 6-panel truss. Hence in this case $wph'/h = 325.8 \div 2.5 = 130.3$. If the stresses in the other diagonals had been based on the same uniform load the coefficients found by dividing each stress by 130.3 would agree with those in the table for the conventional or exact method according to which of these methods had been used. For reasons stated in the preceding article, the coefficients derived by dividing the stresses based on the concentrated load system by 130.3 should be intermediate between the two sets of values in the table. Thus dividing 216.4, 126.4, 59.8 and 16.9 by 130.3, the resulting coefficients are 1.66, 0.97, 0.46 and 0.13 respectively. Since these values are intermediate between those in the table for a six-panel truss, the stresses in question are proven to be at least approximately correct.

CHAPTER VIII

STRESSES IN BRIDGE TRUSSES WITH INCLINED CHORDS

128. General Considerations. The stresses in both chord and web members of bridge trusses with inclined chords may be conveniently found by the method of moments. That method, applied numerically with the aid of a moment table for concentrated loads and a slide rule, is usually perferred in practice to graphic methods. In the analysis of trusses of this class graphic methods are less frequently used, because (1) they are less rapid, especially if the diagrams are carefully executed to a large scale, and (2) the stresses cannot be determined to the same degree of accuracy as by numeric operations. The second consideration is of little importance, since by the selection of a suitable scale the stresses may readily be found with sufficient accuracy for all practical purposes. Algebraic and graphic methods will both be presented in this and succeeding chapters.

129. Chord Stresses in General. The chord stresses in trusses with inclined chords, both from dead and live loads, may be found by the method of moments in precisely the same manner as for trusses with horizontal chords, as explained for dead loads in Art. 85, for uniform live loading in Art. 100 (First Method), and for concentrated loads in Art. 111 (*c*), except that the lever arm by which the resultant moment is finally divided is affected by the inclination of the chord member for which the stress is to be found.

(*a*) *Pratt Truss with Inclined Chord.* In Fig. 1, taking section 1, the moment centers of *cd* and *BC* are seen to lie at *C* and *c*, respectively, i.e., in the same vertical line. Since the external forces are all vertical these stresses therefore become maximum for the same position of the loading. This is also apparent from $\Sigma H=0$ applied to the forces to either side of the section. Disregarding counter *Dc*, in which the stress is zero when the live load extends over the entire bridge, as required for maximum chord stresses, the horizontal component of stress BC equals stress cd. Hence letting $\overline{bc}=p$, $\overline{BC}=p'$ and M_C=resultant moment about C of the forces to either side of the section:

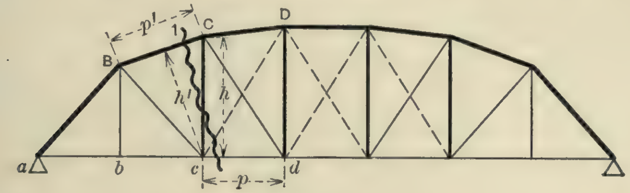


FIG. 1.

$$\text{Stress } cd=M_C/h,$$

and

$$\text{Stress } BC=\text{stress } cd\times p'/p=\frac{M_C}{h}\frac{p'}{p}. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Letting h' denote the lever arm of BC with respect to c, stress BC is also equal to M_c/h'. But since $h'=h\times p/p'$, $M_c/h'=M_cp'/hp$, and since $M_c=M_C$, this agrees with Eq. (1), as it should.

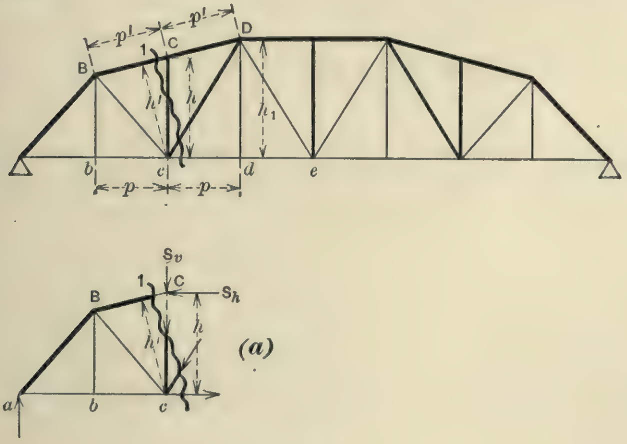


FIG. 2.

(b) *Triangular Truss with Inclined Chord.* Considering stress $BC(=CD)$, Fig. 2, take section 1. The segment of the truss to the left of the section is subject to the forces shown in Fig.

(*a*), and any loads (not shown) that may exist between *a* and *c*. Stress *BC* may be found, as in the previous case, directly by moments about *c*, i.e., stress $BC = M_c/h' = M_c/h \times p'/p$; or the analysis may be made as follows:

At *C*, Fig. (*a*), resolve stress *BC* into horizontal and vertical components, S_h and S_v. Taking moments about *c*, $S_h = M_c/h$ and stress $BC(=CD) = S_h \times p'/p = M_c/h \times p'/p$, as before.

Evidently, stress $cd(=de) = M_D \div h_1$, as for the Pratt truss with horizontal or inclined upper chord.

(*c*) *Trusses with Both Chords Inclined.* From the foregoing it is obvious that Eq. (1) is also applicable to trusses in which both chords are inclined, *h* denoting always the vertical distance from the moment center to the chord member in question, and p'/p the ratio of the length of that member to its horizontal projection.

Chord stresses will not be further considered in this chapter except in connection with examples.

130. Web Stresses in General. In trusses with inclined chords the shear, *V*, in any panel is equal to the algebraic sum of the vertical components of the stresses in the main diagonal and the inclined chord member or members in that panel. In general, the depth of truss increases from the ends to the center. The vertical components of the stresses in the chords and the main diagonal then act in the same direction, except for counter stresses in counterbraced diagonals. Thus in Fig. 3 let *S* and *S′* denote the stresses in *Cd* and *CD* respectively, and let it be assumed that the loading is such that *S* is tensile, as shown in the figure. Resolve *S* into horizontal and vertical components at *d*, and *S′* at *G*, so that the moments of their horizontal components about

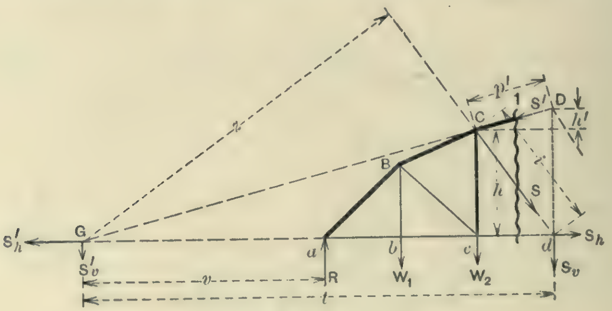


FIG. 3.

d become zero. Let M_d equal the moment of R, W_1 and W_2 about d—i.e., the bending moment at d; then $S_v' = M_d/t$. If V is the shear, $R - W_1 - W_2$, in panel cd,

$$S_v = V - S_v',$$

whence

$$S = (V - S_v')\frac{t}{h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or,

$$S = \left(V - \frac{M_d}{t}\right)\frac{i}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Letting z denote the lever arm of Cd with respect to G, from similar triangles, $i/h = t/z$, so that Eq. (3) may be written:

$$S = V\frac{i}{h} - \frac{M_d}{z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If both chords are horizontal, z = infinity, and $S = Vi/h$, as it should. The term M_d/z therefore denotes the diminution of the stress in the diagonal due to the inclination of the chord member in the same panel.

With the notation in the figure, $S_v' = S' \times h'/p'$, whence Eq. (2) becomes

$$S = \left(V - S'\frac{h'}{p'}\right)\frac{i}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) may be conveniently applied to finding S if the simultaneous chord stress S' is known. Thus in the computation of dead-load stresses, the chord stresses may first be determined, and thence the stresses in the diagonals. The stresses in the verticals may then be found from $\Sigma V = 0$ applied to the joints in the horizontal chord. Eq. (5) is not well adapted to the computation of live-load stresses, because it requires an additional operation for the determination of S' for the loading in position for maximum S, since that position is not the same as that for maximum S'.

Again, if M_G and M_a denote the moments of R, W_1 and W_2 about G and a, respectively,

$$S = \frac{M_G}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which $M_G = M_a - (R - W_1 - W_2)v = M_a - Vv$. For the usual case of tension in Cd the moment M_G must be counter-clockwise; therefore, since M_a is clockwise, the moment Vv must be opposite in character to M_a and greater in magnitude. Hence, for this usual case $M_G = Vv - M_a$ and Eq. (6) may then be written:

$$S = \frac{Vv - M_a}{z}. \qquad (7)$$

Since $z = th/i$, Eqs. (6) and (7) may be changed to the following forms:

$$S = \frac{M_G}{t}\frac{i}{h}, \qquad (8)$$

and

$$S = \frac{Vv - M_a}{t}\frac{i}{h}. \qquad (9)$$

While these equations apparently avoid the necessity of computing the lever arm z, they contain the algebraic expression for that lever arm, so that they are neither more nor less convenient than the previous equations.

Eqs. (6) and (8) are best adapted to live-load stresses, and Eqs. (7) and (9) to dead-load stresses. The latter are more convenient than Eq. (5), as will be seen hereafter (Art. 132, Example).

If both chords are horizontal, v=infinity. As v approaches infinity, the ratio of v to t, as well as the ratio of the lever arms of W_1 and W_2 about G, to t, approaches unity and M_G/t approaches $R - W_1 - W_2 = V$. Therefore, when both chords are horizontal, Eq. (8) becomes $S = Vi/h$, as it should.

131. Lever Arms in Terms of Panel Lengths. In computing the stresses in trusses with inclined chords, it is convenient, as will be seen later, to express certain lever arms in terms of panel lengths. Suitable formulas in that connection will now be derived.

The following notation is used in Figs. 4 (*a*), (*b*), and (*c*):

i = length of the diagonal whose lever arm about G is z,
h = vertical projection of the diagonal,
t = distance from G to the intersection of the diagonal with the horizontal chord.

Then from the known dimensions, h and h_1, of the truss, expressed in feet, the distance t in terms of panel lengths is as follows:

In Fig. (*a*) $$t=\frac{h_1}{h_1-h}.$$

In Figs. (*b*) and (*c*), $$t=\frac{h_1}{h-h_1}.$$

In general, from similar triangles,

$$z=t\frac{h}{i}, \quad \ldots\ldots\ldots \quad (10)$$

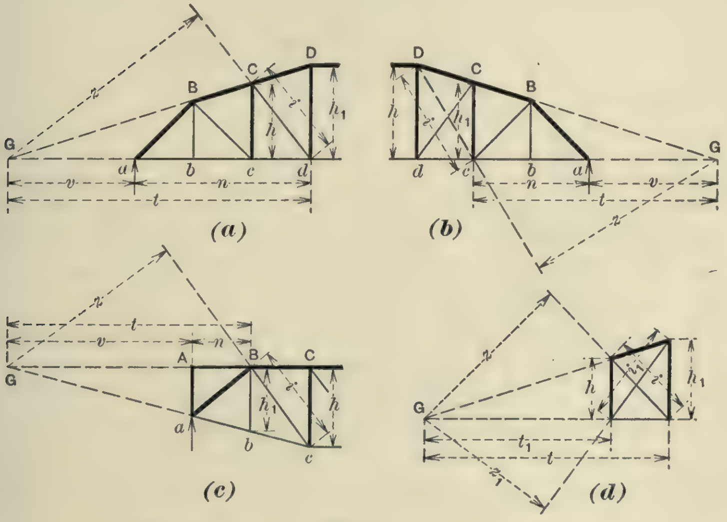


FIG. 4.

from which z may be found in terms of panel lengths by expressing t in like units and h and i in feet. Again, in Figs. (*a*), (*b*), and (*c*),

$$v=t-n, \quad \ldots\ldots\ldots \quad (11)$$

from which v may be found in terms of panel lengths by expressing t and n in panel lengths.

For two diagonals in the same panel, the lever arms of the diagonals are in the inverse ratio of the lengths of the members. Thus, in Fig. (*d*), $z=th/i$ and $z_1=t_1h_1/i_1$. But $th=t_1h_1$; therefore $z/z_1=i_1/i$.

The general case in which both chords are inclined at unequal angles with the horizontal is shown in Fig. 5. Eq. (10) is applicable also to this case, t denoting the horizontal distance from G to the diagonal. It is more convenient, however, under these conditions to express the value of z in another form. With the notation shown in the figure, k denoting the normal distance from d to CD, the following relations may be derived from similar triangles:

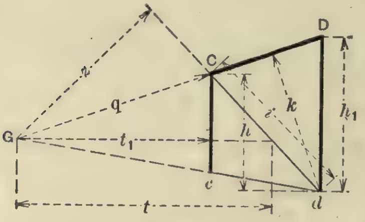


FIG. 5.

$$\frac{t_1}{q}=\frac{k}{h_1} \quad \text{and} \quad \frac{z}{q}=\frac{k}{i},$$

whence,

$$z=t_1\frac{h_1}{i}. \qquad (12)$$

As will be seen later, this equation may be conveniently applied to bridge trusses in which both chords are inclined, (Art. 145, Example) and to the trussed supports ('bents') of viaducts (Chapter XV).

If the heights h and h_1 in any of the above figures are varied in a constant ratio, it is seen that the position of the moment center G remains unaffected. Since the horizontal lever arms v, t, t_1, etc., remain unchanged, the stresses in the verticals are unaltered, as will be seen in the next article. The principle that *the stresses in the vertical members of a truss are not affected if all its vertical dimensions are varied in a constant ratio* is a perfectly general one. The lengths of the inclined lever arms, z and z_1, and therefore the stresses in the diagonals, are obviously affected by variations in h and h_1. But since under the conditions stated, t in Eq. (3), Art. 130, is constant, *the stress in any diagonal varies in proportion to the change in the secant, (i/h) of the angle which it makes with the vertical.*

132. Pratt Truss with Inclined Chords. Dead-load Stresses. (*a*) *Diagonals.* In Fig. 6, consider the stress S_d in any diagonal, as Cd, for example. Take section 1, and let M denote the moment of all forces to the left of the section about G, the intersection of CD and cd. Let $W=W_1+W'$ and let n denote the number of apex loads to the left. Then if u equals the distance from G to the center of gravity of the loads:

$$S_d = \frac{Rv - nWu}{z}, \quad \ldots \ldots \quad (13)$$

in which it is convenient to express v, u and z in terms of panel lengths. The stresses in the diagonals may be readily evaluated by this equation, which is a special form of the general equation (6), Art. 130. If N denotes the number of panels in the truss, $R = \frac{1}{2}(N-1)W$.

If both chords are horizontal, v = infinity. As v approaches infinity, both v/z and u/z approach t/z, which by Eq. (10) equals i/h. Therefore, when both chords are horizontal, Eq. (13) becomes

$$S_d = (R - nW)\frac{i}{h},$$

in which $R - nW$ denotes the shear in the panel and i/h the secant of the angle between the diagonal and a vertical.

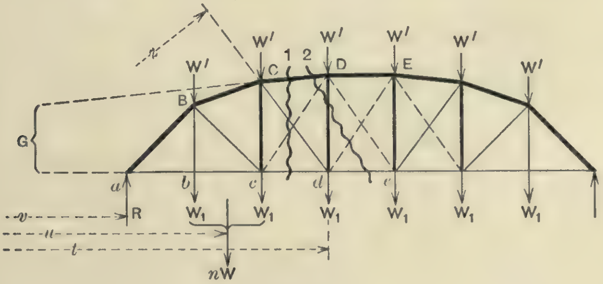


FIG. 6.

(b) *Verticals.* Considering the stress in post Dd and taking section 2, the moment center G is seen to coincide with that for the diagonal, Cd, which meets the post in the *straight* chord. The forces to the left of section 2 are the same as those to the left of section 1, except that W_1 at d is now added, thus increasing the negative moment in Eq. (13) by W_1t. But since t is also the lever arm of stress Dd, the latter may conveniently be expressed as follows:

$$S_p = \frac{Rv - nWu}{t} - W_1. \quad \ldots \ldots \quad (14)$$

Thus the values $Rv-nWu$ previously computed, in a given case, for the stresses S_d in the diagonals may be utilized in finding the stresses S_p in the posts.

Eq. (14) might have been derived directly from Eq. (13) by analyzing the forces at joint d. If $\overline{Cd}=i$ and $\overline{Cc}=h$:

$$S_p=S_d\frac{h}{i}-W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which $h/i=z/t$. Eq. (15) is applicable also to trusses with horizontal chords.

As will be seen in the example at the end of this article, the dead-load web stresses may be found somewhat more conveniently by Eq. (7) Art. 130, by which,

$$S_d=\frac{Vv-M_a}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

whence

$$S_p=\frac{Vv-M_a}{t}-W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

in which V is the shear, $R-nW$, in the panel containing the diagonal, and M_a the moment of nW about the left support. The lever arms v, t and that of nW may be conveniently expressed in terms of panel lengths.

(*c*) *Chords.* The stresses in the chords may be computed as explained in Art. 129.

Example. Required the dead-load stresses in the trusses of the 200-ft., single-track railroad bridge shown in Fig. 7, from the following loads per foot of bridge:

Steel	Trusses and Bracing,	1,875 lbs.
	Floor beams and Stringers,	625 "
	Total,	2,500 "
	Track,	400 "
	Grand total,	2,900 "

Solution. All loads and stresses will be expressed in thousands of pounds. The weights of the steel floor system and track will be assumed to be applied wholly at the lower chords. The remaining weight will be equally divided between the four chords.

Apex Loads:

$$\text{Lower,}\quad\left(\frac{0.625+0.4}{2}+\frac{1.875}{4}\right)28.57=28.0.$$

$$\text{Upper,}\quad\frac{1.875}{4}\times 28.57 \qquad\qquad =13.4$$

$$\text{Total,}\qquad =41.4$$

Check: $\frac{2.9}{2}\times 28.57=41.4.$

$$\text{End Reaction}=41.4\times 3=124.2.$$

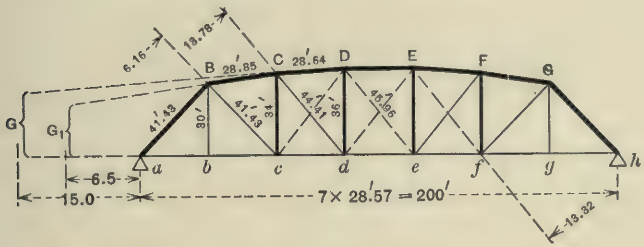


FIG. 7.

Lever arms in terms of panel lengths (Eq. (10), Art. 131):

$$\frac{30}{34-30}-1=6.5;\quad \frac{34}{36-34}-2=15.0.$$

$$Bc,\ (6.5+2)\frac{30}{41.43}=6.16;\quad Cd,\ (15+3)\frac{34}{44.41}=13.78.$$

Diagonals:

$$aB=124.2\times\frac{41.43}{30}=(-)171.5.$$

First Method. Eq. (13),

$$Bc=[(124.2\times 6.5)-(41.4\times 7.5)=496.8]\div 6.16=(+)80.7;$$

$$Cd=[(124.2\times 15)-(41.4\times 2\times 16.5)=496.8]\div 13.78=(+)36\ 1;$$

$$De=0.$$

Second Method. Eq. (16),

$$Bc=[41.4(\overline{2\times 6.5}-1)=496.8]\div 6.16=(+)80.7;$$

$$Cd=[41.4(\overline{1\times 15}-3)=496.8]\div 13.78=(+)36.1.$$

The second method is seen to be somewhat simpler than the first.

Third Method. Eq. (5), Art. 130. By this method, the stresses in the diagonals may be derived from the chord stresses (see below), viz.:

$$Bc=\left[(41.4\times2)-175.6\times\frac{4}{28.85}=58.4\right]\frac{41.43}{30}=(+)80.7;$$

$$Cd=\left[(41.4\times1)-197.6\times\frac{2}{28.64}=27.6\right]\frac{44.41}{34}=(+)36.1.$$

Although by this method the lever arms of the diagonals need not be found, it is less convenient than the second method, and it is open to the objection that the web stresses are not computed independently, and that they are affected by errors in the chord stresses. Again, for reasons stated in Art. 130, this method is not well adapted to live-load stresses, and it is, generally speaking, desirable to use the same method for both dead- and live-load stresses.

Verticals.

$$Bb=+28.0.$$

By Eq. (14) or (17),

$$Cc=496.8\text{ (see }Bc)\div 8.5-28=(-)\ 30.4;$$

$$Dd=496.8\text{ (see }Cd)\div18.0-28=(+)\ 0.4.$$

If the stresses in the diagonals are determined by the third method, the stresses in the verticals may be found as follows from the analysis of the vertical forces at c and d:

$$Cc=58.4\text{ (see }Bc)-28.0=(-)30.4;$$

$$Dd=27.6\text{ (see }Cd)-28.0=(+)0.4.$$

Chords.

By Eq. (6), Art. 85,

$$ab(=bc)=\frac{1\times6}{30}\times\frac{41.4\times28.57}{2}=(+)118.3;$$

$$cd=\frac{2\times5}{34}\times\quad\text{“}\quad=(+)173.9;$$

$$de=\frac{3\times4}{36}\times\quad\text{“}\quad=(+)197.1;$$

By Eq. (1), Art. 129,

$$BC=173.9\times\frac{28.85}{28.57}=(-)175.6;$$

$$CD=197.1\times\frac{28.64}{28.57}=(-)197.6;$$

$$DE=-de=(-)197.1.$$

From an analysis of the forces at joint *D*, it is seen that stress *Dd* is equal to the difference between the vertical (upward) component of stress *CD* and the panel load, 13.4, at *D*, i.e.,

$$Dd = 197.6 \times \frac{2}{28.64} - 13.4 = (+)0.4.$$

This agrees with stress *Dd* as previously obtained, and serves also as a check on the maximum chord stress.

133. Graphic Method. The solution of the example in the preceding article by the graphic method is shown in Fig. 8. In the original drawing the truss diagram, Fig. (*a*), was drawn to a scale of 1 in. = 30 ft. and the force diagram, Fig. (*b*), to a scale of 1 in. = 4,000 lbs.

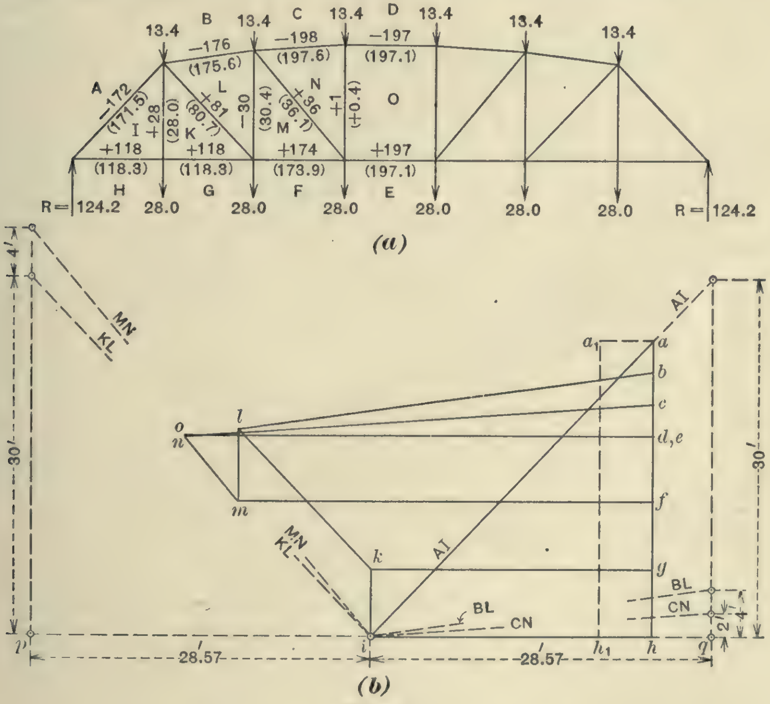


FIG. 8.

The resulting stresses are the upper values in Fig. (*a*). The lower values, in parentheses, are those previously found by computation. The accuracy of the results obtained graphically in this case could hardly have been realized if Fig. (*b*) had been drawn in the usual manner by laying off the lines parallel to the comparatively short guide lines in Fig. (*a*). For accuracy it is important on the one hand that the truss diagram should be drawn to a liberal scale, but on the other hand this renders the transference of directions to the force diagram less convenient and adds to the liability of error incident to the shifting of triangles through long distances. In a case such as this, where the coordinates of all the sloping lines in the truss are known, it is advisable to lay off the panel lengths *ip* and *iq* in Fig. (*b*) to a large scale, and then to lay off on the verticals through *i* and *q* the known vertical projections of all sloping lines. The line *AI* and the broken lines radiating from *i* may then be drawn as hair lines, and used as guide lines in the transference of directions in the force diagram. It is, in fact, not necessary to draw the broken lines, but merely to locate their extremities and distinguish them by dots in small circles as in the figure. The construction of the force diagram is begun by laying off the reaction R to a suitable scale from some point on *iq*, as h_1, for example, and drawing a_1a horizontally to its intersection with *ai*. The successive apex loads *ab*, *bc*, etc., on the left half of the truss are then laid off on *ah*, and the diagram continued in the usual manner. This construction dispenses with the need of an accurately drawn truss diagram.

This example serves to illustrate one of the disadvantages of the graphic method referred to in Art. 61. According to the previously computed values, stress *no* is $+0.4$ and the greatest chord stress, *cn*, is -197.6, the latter being nearly 500 times as great as the former. By any reasonable scale, the line *no*, Fig. (*b*), is so short, that unless the diagram is constructed with extraordinary care the point *n* may be found to lie above *o*, indicating compression instead of tension in this member, aside from the uncertainty as to the magnitude of the stress. It is to be observed, however, that inaccuracies from this cause are as a rule of little or no practical importance, since (1) members subjected to very small stresses are given a much larger sectional area than that required for the computed stresses, and (2) such members are usually counterbraced.

134. Live-load Stresses from Uniform Loading by the Conventional Method. By the conventional method of treating uniform loading in the analysis of web stresses, full panel loads are assumed to be applied at successive joints according to the requirements for maximum stress in the given member, but the existence of the forward half-panel load is ignored (Art. 98).

(*a*) *Diagonals.* Referring to Fig. 9, consider diagonal *Cd*. A load at any joint to the left of section 1, as at *c*, for example, Fig. (*a*), will cause the resultant moment about *G* of the load and its reaction at *a* to be clockwise, indicating compression in *Cd*. If a load is placed at any joint to the right of the section the moment of its reaction at *a* (the only force now to the left of the section) about *G* will be counter-clockwise, indicating tension in *Cd*. It is seen, therefore, that for maxmium tension in *Cd*, the joints to the right of the section must be fully loaded, as shown in the figure. Letting *v* denote the distance from *a* to *G*, the stress in the diagonal is

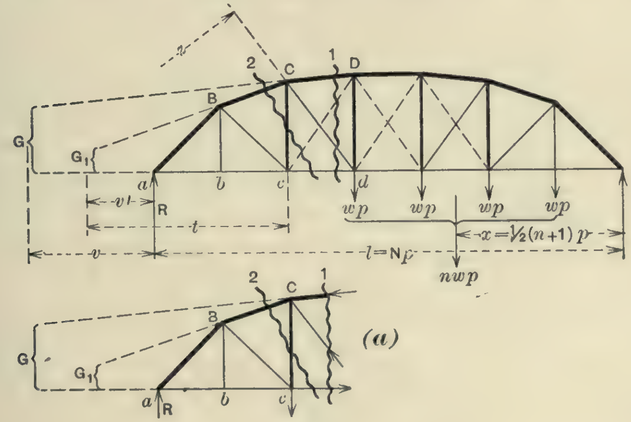


FIG. 9.

$$S_d = R\frac{v}{z}. \qquad \qquad (18)$$

If *n* denotes the number of panels to the right of the section and *N* the number in the truss, $R = wp(1+2+\ .\ .\ .\ .\ .\ +n)/N$. The sum of the terms in the parentheses is $\frac{1}{2}n(n+1)$, whence

$$S_d = \frac{n(n+1)}{2}\frac{wp}{N}\frac{v}{z}, \qquad \qquad (19)$$

in which wp/N is a constant in any given case, and v and z may be conveniently expressed in panel lengths.

Since n denotes also the number of panel loads wp, $R=nwp\,x/l$, whence Eq. (18) becomes

$$S_d=nwp\frac{x}{l}\frac{v}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

in which x and l may be conveniently expressed in panel lengths in any given case. If this is done algebraically, Eq. (20) is reduced to the same form as Eq. (19).

(*b*) *Verticals.* Considering the stress in post Cc, Fig. (*a*), from a load W at c, and taking section 2, it is seen from the moment of R and W about G_1 that stress Cc is tensile. For maximum compression in Cc loads wp must be applied at all joints to the right of c. This condition of loading is the same as that previously found for maximum tension in Cd. Thus it is seen that *the stresses in a diagonal and a post meeting in the unloaded chord become maximum concurrently.* This is true in general only for live-load stresses based on the conventional method. If the stresses are determined by the exact method for uniform loading or for concentrated loads the relation stated holds true only if the chord members on either side of the point of intersection have the same slope, as will be shown hereafter (Art. 136).

The stress in post Cc is

$$S_p=R\frac{v'}{t}=\frac{n(n+1)}{2}\frac{wp}{N}\frac{v'}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Thus the values $R=\frac{n(n+1)}{2}\cdot\frac{wp}{N}$, previously computed in any given case for the stresses S_d in the diagonals, may be utilized in finding the stresses S_p in the posts.

If the chord members on either side of the post Cc have the same slope, G_1 coincides with G. In that case $v=v'$ and, from Eqs. (19) and (21), $S_p=S_d\,z/t$.

(*c*) *Counters.* Referring to Fig. 10, for maximum tensile stress in counter Dc, for example, panel loads wp must be assumed to act at joints b and c only.

As before, v denotes the distance from the left end reaction to the moment center G, and Eq. (18) is therefore applicable to this case after replacing z by z_1. The stress in the counter, if the main diagonal *Cd* in the same panel is assumed to be omitted, is

$$S_c = R\frac{v}{z_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

In reality the diagonal *Cd* not only exists, but it is subject to a dead-load tensile stress S_D. As the live load comes upon the bridge from the right end, it produces compression in *Cd* which that member is capable of developing up to the limit S_D. This is equivalent to a moment $S_D z$ about *G*. Under ideal adjustment the counter *cD* will not come into play until the tensile dead-load stress in *Cd* has been exhausted, and that member, being flexible, is on the point of buckling. The moment *Rv* from which the stress in the counter, S_c, was derived should therefore be diminished by $S_D z$. Thus

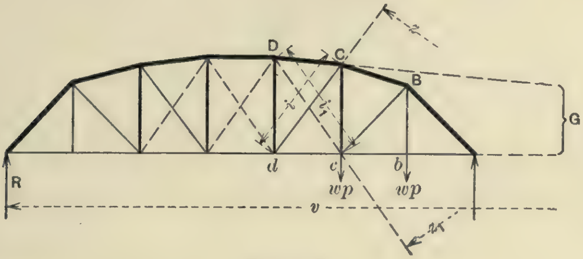


FIG. 10.

$$S_c = \frac{Rv - S_D z}{z_1},$$

in which $S_d z/z_1$ may also be regarded as the dead-load compressive stress in the counter if the main diagonal in that panel did not exist. But for reasons stated in Art. 102 it is customary to apply a reduction factor of ½ to dead-load stresses in estimating their effectiveness in counteracting live-load stresses. Also, $z/z_1 = i_1/i$, (Art. 131), whence

$$S_c = R\frac{v}{z_1} - \tfrac{1}{2}S_D\frac{i_1}{i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

The maximum compressive counter stress in the main diagonal *Bc* is never sufficiently large to neutralize the dead-load tensile stress in that member. If the method to be used in proportioning the members requires the computation of minimum stresses (Art. 101), the maximum live-load compressive stresses in main diagonals in panels containing no counters may be found by Eq. (22) and subtracted from the dead-load tensile stresses.

(*d*) *Chords.* The stresses in the chords may be found as explained in Art. 129.

Example. Required the maximum live-load stresses in the trusses of the 200-ft. single-track railroad bridge shown in Fig. 11, from a uniform load of 7400 lbs. per foot of bridge, by the conventional method.

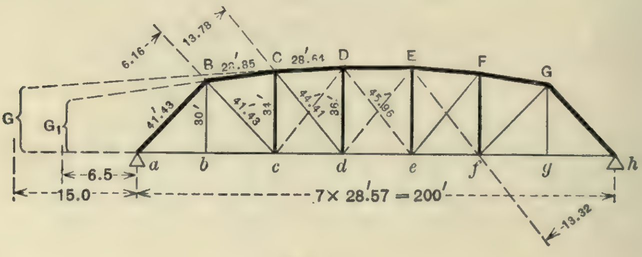


FIG. 11.

Solution. All loads and stresses will be expressed in thousands of pounds. The values of the lever arms shown in the figure were computed in the example in Art. 132.

Panel Load:
$$wp=\frac{7.4}{2}\times 28.57=105.7.$$

Diagonals:
$$aB=[105.7\times 3=317.1]\times\frac{41.43}{30}=(-)437.8.$$

Stresses *Bc* and *Cd* may be found by Eq. (19). For *De* the factor v/z in that equation must be replaced by the secant of the angle between *De* and a vertical, since the chords in the middle panel are horizontal. For the counter *Ef*, v/z becomes v/z_1, Eq. (22). Thus

$$Bc=\left[\frac{5(5+1)}{2}\times\frac{105.7}{7}=226.5\right]\times\frac{6.5}{6.16}=(+)239.0;$$

$$Cd=\left[\frac{4(4+1)}{2}\times\frac{105.7}{7}=151.0\right]\times\frac{15}{13.78}=(+)164.4;$$

$$De=\left[\frac{3(3+1)}{2}\times\frac{105.7}{7}=\ 90.6\right]\times\frac{45.96}{36}=(+)115.7;$$

$$Ef=\left[\frac{2(2+1)}{2}\times\frac{105.7}{7}=\ 45.3\right]\times\frac{7+15}{13.32}=(+)\ 74.8.$$

The stress in counter Ef must be reduced in accordance with Eq. (23) to allow for the relief afforded by the dead-load stress $Fe(=Cd)$ in the main diagonal in the same panel. From the example in Art. 132, that stress is $+36.1$. Hence

$$Ef=74.8-\left(\frac{36.1}{2}\times\frac{45.96}{44.41}=18.7\right)=(+)56.1.$$

If the members are to be proportioned by the method involving the minimum as well as the maximum stress in each member (Art. 101), the counter stress in Gf will have to be found. This is greatest for a panel load 105.7 at g, for which $R=105.7\times\frac{1}{7}=15.1$. Hence

$$\text{Counter stress } Gf=15.1\times\frac{7+6.5}{6.16}=(-)33.1.$$

The dead-load stress in the corresponding member Bc was found to be $+80.7$ (Art. 132), so that the minimum stress is $+80.7-33.1=(+)47.6$.

The minimum stress in the counters and in the main diagonals in panels containing counters is zero.

Verticals:

$$Bb=(+)105.7.$$

By Eq. (21),

$$Cc=151.0\ (\text{see } Cd)\frac{6.5}{8.5}=(-)115.4;$$

$$Dd=\ 90.6\ (\text{see } De)\ \frac{15}{18}=(-)\ 75.5.$$

The maximum live-load tensile stresses in the posts will be considered later (Art. 141). This tensile stress is also the ' minimum ' stress in these members in the sense in which that term is used in considering the maximum *range* of stress in a member (Art. 101).

Chords. For maximum chord stresses a panel load 105.7 must be assumed at every joint of the lower chord. The dead-load chord stresses were computed for a panel load of 41.4 (Art. 132); hence it is only necessary to multiply these stresses by 105.7/41.4, whence,

$$ab(=bc)=+302.1;\qquad BC=-448.4;$$

$$cd=+444.0;\qquad CD=-504.5.$$

$$de=+503.2.\qquad DE=-503.2.$$

135. Graphic Method. The example in the preceding article will now be solved by the graphic method. For maximum stress in all chord members and in the inclined end post a live panel load, 105.7, must be applied at every joint of the loaded chord. These stresses may be found graphically by a single diagram, as in Fig. 8, Art. 133; or better, numerically by multiplying the dead-load stresses from the latter by the ratio of the live and dead-load panel concentrations, i.e., 105.7/(28+13.4).

The stress in the hip vertical, *Bb*, is +105.7. The live-load stresses in the remaining web members become maximum for different conditions of loading, so that these stresses cannot be found directly by a single diagram. For the load advancing from the right, however, and disposed for maximum stress or counter stress in successive web members, it has been seen that by the conventional method it is assumed that no loads exist to the left of the right end of a diagonal, nor to the left or at the foot of a vertical. Thus, for example, for maximum stress in *Cc* and *Cd*, Fig. 12, no loads occur at *b* and *c*. If therefore the left reaction *R* is assumed to be unity, and the corresponding stresses in *Cc* and *Cd* are found graphically by working from the left end of the truss, the true stresses in these members are the products of the stresses for $R=1$ and the true reaction *R* for the loading in position for maximum stress in *Cc* and *Cd*.

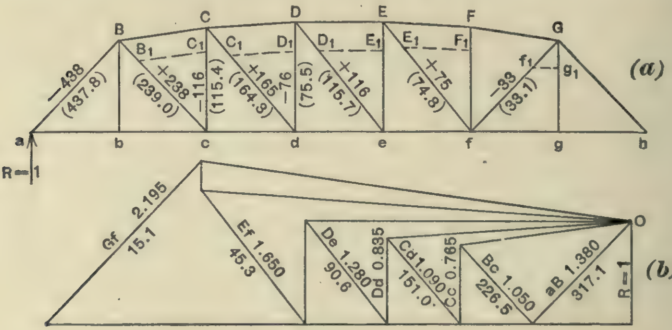


FIG. 12.

In Fig. (*b*) the upper figures on the diagonals denote the values of the stresses for $R=1$ as scaled from that diagram (drawn originally to a scale of 1 in. = 0.6 and scaled to the nearest 0.005) and the lower figures are the reactions, R, in thousands of pounds, when the load is in position for maximum stress in each member. These values of R are given in the example in the preceding article. The product of each pair of values in Fig. (*b*) is the stress expressed by the upper figures in Fig. (*a*). For the verticals, the values of the stresses for $R=1$ are given in Fig. (*b*). The reaction to be used for the maximum stress in any vertical is the same as that for the diagonal which meets that vertical in the unloaded chord. The lower figures in parentheses in Fig. (*a*) are the values obtained by computation in the preceding article. The close agreement between these values could hardly have been realized if the lines radiating from O (Fig. *b*) had not been drawn independently of Fig. (*a*) by laying off the coordinates of the corresponding chord members to an enlarged scale from O, as in Fig. 8, Art. 133.

136. Live-load Stresses from Concentrated Loading. (*a*) *Main Diagonals.* Referring to Fig. 13, let the stress in main diagonal Cd be considered. Take section 1. The moment center lies at G. Place the critical wheel at d. Then, letting M_G denote the moment of the forces to the left of the section about G,

$$M_G=\frac{Wx}{l}v-\frac{W_1x_1}{p}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For maximum M_G,

$$\frac{dM_G}{dx}=\frac{Wv}{l}-\frac{W_1u}{p}=0,$$

or

$$W=\frac{u}{v}\frac{l}{p}W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which is the criterion for maximum stress.

If the truss has N equal panels, the criterion becomes

$$W=\frac{u}{v}NW_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which v and u may be conveniently expressed in panel lengths. For parallel chords, $v=$infinity. As v approaches infinity u/v approaches unity. Therefore, when both chords are horizontal, $W=NW_1$, as previously found (Art. 111).

Since u/v is an improper fraction, the coefficient of W_1 in the criterion is greater for a truss with inclined chords than for one with horizontal chords having the same number of panels. Hence, if the span and number of panels are alike in both cases, the criterion for corresponding panels is satisfied for *less* loading, W_1, in the panel in question for the truss with *inclined* chords.

The position which satisfies the criterion having been found with the aid of the moment table as explained in Art. 113, the corresponding stress is M_G/z. In Eq. (24) let $R=Wx/l$ and $r=W_1x_1/p$. The stress in the diagonal may then be expressed as follows:

$$S_d=\frac{Rv-ru}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

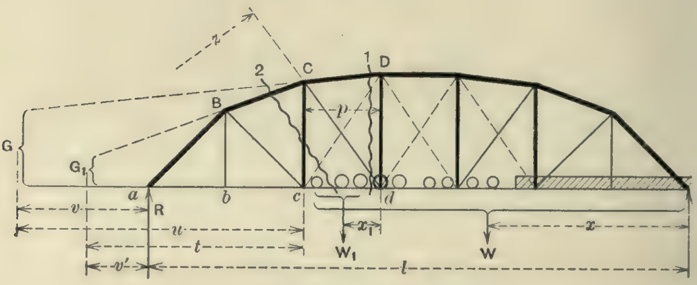


FIG. 13.

in which R and r are readily found with the aid of a moment table (Art. 39). The lever arms v, u and z may be conveniently expressed in panel lengths.

(*b*) *Posts.* Let the stress in post *Cc* be considered. Taking section 2, the moment center lies at G_1, the intersection of *BC* and *cd*. If the critical wheel is again placed at *d* (as for the diagonal *Cd* which intersects the post in the *unloaded* chord), the moment of the forces to the left of the section about G_1 will be expressed by Eq. (24), after the lever arms v' and t are substituted for v and u respectively. The criterion for maximum stress in the post may therefore be obtained by making the same substitutions in Eq. (25) or Eq. (26), whence

$$W=\frac{t}{v'}\frac{l}{p}W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

or, for a truss of N equal panels,

$$W=\frac{t}{v'}NW_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Since the criteria for Cc and Cd are different, the stresses in these members usually become maximum for different wheels at the critical point d. If, however, the chord members to either side of C have the same slope, $v=v'$ and $u=t$, and the criterion for Cc has then the same numeric value as that for Cd. In that case, therefore, the stresses in a post and a diagonal meeting in the *unloaded* chord become maximum concurrently.

By analogy with Eq. (27) the stress in the post is $(Rv'-rt)/t$, or, in general,

$$S_p=R\frac{v'}{t}-r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Eqs. (27) and (30) may be expressed as follows, c and c_1 being constants:

$$S=cR-c_1r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For the diagonals, $c=v/z$ and $c_1=u/z$.

For the posts, $c=v'/t$ and $c_1=1$.

The values R and r may be obtained with the aid of a moment table for the load in position for maximum stress in each web member, and the stresses may then be computed by the general equation (31).

The maximum tensile counter stresses in the vertical posts will be considered later (Art. 141).

(*c*) *Counters.* Referring to Fig. 14, consider the stress in counter Dc, assuming at first that the main diagonal Cd in the same panel is omitted. Take section 1. The moment center lies at G. Place the critical wheel at c. Then letting M_G denote the moment of the forces to the left of the section about G,

$$M_G=\frac{Wx}{l}v-\frac{W_1x_1}{p}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

For maximum M_G,

$$\frac{dM_G}{dx}=\frac{Wv}{l}-\frac{W_1u}{p}=0,$$

$$W=\frac{u}{v}\frac{l}{p}W_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

This criterion is seen to be of precisely the same form as that for the main diagonals, Eq. (25). In both equations v denotes the lever arm of the left reaction about G, and u the lever arm of the forward panel load about that point. But for the counters u/v is a proper fraction, so that the coefficient of W_1 in the criterion is less for a truss with inclined chords than for one with horizontal chords having the same number of panels. Hence, if the span and number of panels are alike in both cases, the criterion for counters in corresponding panels is satisfied for *greater* loading, W_1, in the panel in question, for the truss with *inclined* chords. For the main diagonals the opposite relation was found to hold true.

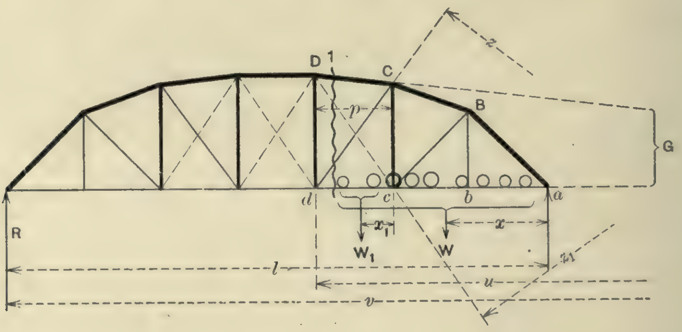


Fig. 14.

If the truss has N equal panels, the criterion becomes

$$W = \frac{u}{v} N W_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

After finding the position which satisfies this criterion the corresponding stress may be obtained by Eq. (27) or (31) after substituting z_1 for z, as in Art. 134 (c). The resulting stress should be reduced by a $\frac{1}{2}S_D i_1/i$ in which S_D is the dead-load stress in the main diagonal in the same panel as the counter, and i and i_1 are the lengths of the main diagonal and counter, respectively.

The maximum compressive counter stress in main diagonals in panels containing no counters may be found by determining the position for maximum stress from the criterion for counters and then deriving the stress by Eq. (27), in which v and u have now the same general significance as in Fig. 14.

(*d*) *Chords.* The stresses in the chords may be found as explained in Art. 129.

Example. Required the maximum live-load stresses in the trusses of the 200-ft. single-track railroad bridge, Fig. 15, from Cooper's E–60 loading.

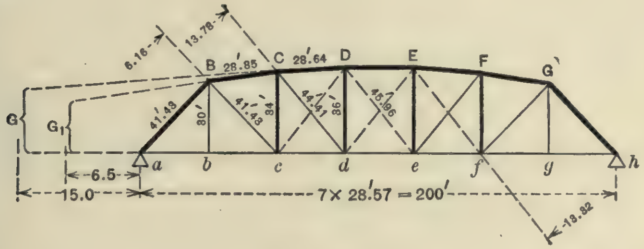


FIG. 15.

Solution. All loads and stresses will be expressed in thousands of pounds. The values of the lever arms shown in the figure were computed in the example in Art. 132.

Diagonals. The criteria for maximum stress, and the positions satisfying the same (found with the aid of the moment table according to Eq. (40), Art. 111, for *aB* and *De,* Eq. (26) for *Bc* and *Cd,* and Eq. (33) for *Ef*) are as follows:

$$aB, \quad W=7W_1 \qquad \text{(Wheel 4 at } b\text{)};$$

$$Bc, \quad W=\frac{7.5}{6.5}\times 7W_1=8.1W_1 \text{(Wheel 3 at } c\text{)};$$

$$Cd, \quad W=\frac{17}{15}\times 7W_1=7.9W_1 \text{ (Wheel 3 at } d\text{)};$$

$$De, \quad W=7W_1 \qquad \text{(Wheel 3 at } e\text{)};$$

$$Ef, \quad W=\frac{18}{22}\times 7W_1=5.7W_1 \text{ (Wheel 3 at } f\text{)};$$

$$\text{(Counter stress) } Gf, \quad W=\frac{8.5}{13.5}\times 7W_1=4.4W_1 \text{(Wheel 2 at } g\text{)}.$$

The values of the left end reaction, *R*, and the forward apex load, *r*, for each of the above positions may be readily found with the aid of the moment table. These values are as follows:

	aB	*Bc*	*Cd*	*De*	*Ef*	*Gf*
$R=$	342.5	239.0	164.2	101.3	52.0	15.8
$r=$	25.2	12.1	12.1	12.1	12.1	4.2

The stresses aB and De may then be obtained from the shears in the corresponding panels, and the stresses in the other diagonals by Eq. (27), viz:

$$aB=(342.5-25.2)\frac{41.43}{30}=(-)438.2;$$

$$Bc=(239.0\times6.5-12.1\times7.5)\div6.16=(+)237.5;$$

$$Cd=(164.2\times15-12.1\times17)\div13.78=(+)163.8;$$

$$De=(101.3-12.1)\frac{45.96}{36}=(+)113.9;$$

$$Ef=(52.0\times22-12.1\times18)\div13.32=(+)69.5;$$

$$Gf=(15.8\times13.5-4.2\times8.5)\div6.16=(-)28.8.$$

The stress in counter Ef must be reduced by 18.7, as explained in the example in Art. 134, on account of the dead-load tensile stress in Fe; hence

$$Ef=+69.5-18.7=(+)50.8.$$

The above compressive stress in Gf is of interest only if the members are to be proportioned by the method involving the minimum as well as the maximum stress in each member (Art. 101). In the example in Art. 132 the dead-load stress in this member was found to be +80.7. The minimum stress is therefore $+80.7-28.8=(+)51.9$.

The minimum stress in the counters and in the main diagonals in panels containing counters is zero.

Verticals. The criteria for maximum stress and the positions satisfying the same, Eq. (29), are as follows:

$$Cc,\quad W=\frac{8.5}{6.5}\times7W_1=9.2W_1 \text{ (Wheel 3 at } d\text{)},$$

$$Dd,\quad W=\frac{18}{15}\times7W_1=8.4W_1 \text{ (Wheel 3 at } e\text{)}.$$

These positions are the same as those above for maximum stress in Cd and De respectively; hence the same values of R and r may be used, whence

$$Cc=\left(164.2\times\frac{6.5}{8.5}\right)-12.1=(-)113.5,$$

$$Dd=\left(101.3\times\frac{15}{18}\right)-12.1=(-)72.3.$$

The maximum live-load tensile stresses in the posts will be considered later (Art. 141). This tensile stress is also the 'minimum' stress, in the sense in which that term is used (Art. 101) in considering the maximum *range* of stress in a member.

The maximum live-load tension in Bb is equal to the maximum floor-beam reaction at b. The criterion, $W=2W_1$, Eq. (35), Art. 45, is satisfied with either of the middle drivers of the second engine (wheel 12 or 13) at b. The corresponding reactions, by Eq. (39), Art. 45, are:

$$\text{(Wheel 12) } Bb=213-\frac{1262+1289}{28.57}=(+)123.7;$$

$$\text{(Wheel 13) } Bb=213-\frac{1227+1280}{28.57}=(+)125.2.$$

The minimum stress in Bb is the dead-load stress.

Chords. Stress $ab(=bc)$ becomes maximum concurrently with stress aB. Hence

$$ab(=bc)=(342.5-25.2)\frac{28.57}{30}=(+)302.2.$$

The criterion for maximum stress in cd, Eq. (43), Art. 111, is satisfied with wheel 8 at c. The corresponding moment about the right end support is 66,150. Hence

$$cd=\left[\left(66{,}150\times\frac{2}{7}\right)-4280\right]\div 34=(+)430.0;$$

$$BC=430.0\times\frac{28.85}{28.57}=(-)434.2.$$

Since the number of panels in the truss is odd the chord stresses in the middle panel must be found with due regard to the character of the shear in that panel as explained in Art. 119.

The criterion for maximum bending moment at d is satisfied with wheel 12 at that point. The moment about the right end support is then 64,470, and the left reaction, $64{,}470/200=322.4$. The shear in panel de may be found by subtracting from the left reaction all loads to the left of e and adding the reaction at e from the loads in panel de, i.e.,

$$\text{Shear in panel } de=322.4-387+\frac{1289}{28.57}=-19.5.$$

Since the shear is negative, counter Ed is in action, and the position therefore fulfills the requirements for maximum stress in DE. Hence

$$DE=\left[\left(64{,}470\times\frac{3}{7}\right)-10{,}060\right]\div 36=(-)488.1;$$

$$CD=488.1\times\frac{28.64}{28.57}=(-)489.3.$$

Let the position for maximum stress in de be sought with the critical wheel at e. The criterion is fulfilled with wheel 14 at that point, but the shear in panel de is then positive, indicating that counter De is in action, whereas the requirements for maximum stress in *de* with the critical wheel at *e* demand that the shear should be negative. Wheel 15 at *e* also satisfies the criterion. Let the shear in panel *de* be investigated for the load thus placed. The moment about the right support is then 58,400 and the left reaction is 292.0. The shear in panel *de* may now be most conveniently found by subtracting from the left reaction all loads to the left of *d*, and the reaction at *d* from the loads in panel *de*, i.e.,

$$\text{Shear in panel } de = 292.0 - 228 - \frac{1980}{28.57} = -5.3.$$

Since the shear is negative the requirements are met and

$$de = \left[\left(58{,}400 \times \frac{4}{7}\right) - 16{,}220\right] \div 36 = (+)476.4.$$

The stress in *de* will now be determined for the loading in the position for zero shear in panel *de* (Art. 119), which must be found by trial in the manner illustrated in the example in the article referred to. Thus it is found that the requirement is met when wheel 14 is 4.7 ft. to the left of *e*. The moment about the right support is then 55,750, and

$$de = \left[55{,}750 \times \frac{4}{7} - (13{,}100 + \overline{348 \times 4.7})\right] \div 36 = (+)475.6.$$

Since the shear in the middle panel is zero, the same value should be obtained by taking moments about *D* instead of *E*, i.e.,

$$de = \left[55{,}750 \times \frac{3}{7} - (5{,}240 + \overline{213 \times 7.1})\right] \div 36 = (+)476.1.$$

The slight difference between this value and that previously found is due to the fact that the values in the moment table are not carried out to a higher degree of accuracy. Both of these values are lower, however, than the stress, +476.4, found by the first method.

The dead-load stresses are the minimum stresses in the chords.

137. Graphic Method. The example in the preceding article will now be solved as far as possible by the graphic method.

(*a*) *Chord Stresses.* The chord stresses from a concentrated load system cannot be found graphically by any convenient method. These stresses may be determined approximately, however, by using a so-called 'equivalent' uniform load. If the latter is so chosen that it will give the same maximum shear in the end panel as the concentrated load system, the errors in the chord stresses will seldom or never exceed 5 or 6 per cent., and will usually be much less. Moreover, these errors are always on the side of safety.

Thus, from the example in the preceding article, the maximum live-load shear in the end panel of a seven-panel, 200-ft. railway bridge truss, from Cooper's E–60 loading, is $342.5-25.2=317.3$, in thousands of pounds. If w denotes the intensity of the uniform load per truss, the maximum shear in the end panel from this loading is $3\times 28.57w=85.71w$. If $85.71w=317.3$, $w=3.702$ per truss, i.e., about 7400 lbs. per foot of bridge. The chord stresses for that loading were computed in the example in Art. 134, and these stresses as well as those based directly on Cooper's E–60 loading are tabulated in Art. 139. It is seen that the errors in the chord stresses from the equivalent uniform load are all on the side of safety, the percentage of error being a little above 3 for all chord members except *de*, for which it is 5.6. The agreement for the stresses in the end lower-chord members is exact, as it should be.

The chord stresses from such an equivalent uniform load may be found graphically by a single diagram as in Fig. 8, Art. 133, or better numerically by multiplying the dead-load stresses from the latter by the ratio of the live and dead-load panel concentrations, i.e. $(28.57\times 3.7)/(28+13.4)$.

(*b*) *Web Stresses.* The maximum live-load web stresses from a concentrated load system may be found graphically by an extension of the method explained in Art. 135 for the stresses from uniform loading by the conventional method. In the latter when the loading is advanced from the right and disposed for maximum stress in a given web member the only force to the left of that member is the live-load reaction R at the left end. In the present case, however, there is also a forward apex load, r. For diagonals this load r occurs at the left end of the panel containing the diagonal, and for verticals it acts at their lower extremities in a through bridge, and at their upper extremities in a deck bridge. The values R and r, as computed in the example in the preceding article, are as follows:

	aB	*Bc*	*Cd* and *Cc*	*De* and *Dd*	*Ef*	*Gf*
$R=$	342.5	239.0	164.2	101.3	52.0	15.8
$r=$	25.2	12.1	12.1	12.1	12.1	4.2

After these values have been computed the corresponding web stresses may be found graphically by determining the stresses from R and r separately, and then combining these values. The stresses from R are found in the manner explained in Art. 135. Thus, referring to Fig. 12 (*b*), the stress in Cd from R is $164.2\times 1.09 = 179$. The stresses from r in the diagonals may best be found graphically by recognizing that the loads r do not affect the stresses in members to their left, except in so far as their influence on R is concerned, which is already accounted for in the values of R. Referring to Fig. 12, Art. 135, the effect on stress Cd of $r(=12.1)$ at c, may therefore be found graphically by laying off 12.1 to a convenient scale from d to D_1, drawing D_1C_1 parallel to DC and scaling $C_1d = -15.0$ as the stress desired. The resultant live-load stress in Cd is then $179-15=164$.

The correctness of this procedure is also apparent from the fact that in so far as the stresses from R and r in the diagonals to which they pertain are concerned, the truss may be regarded as rigidly fixed at the right end, and the stresses referred to may then be found as if the left end of the truss were the free end of a cantilever. The loads r would then evidently have no effect on any members to their left.

The stresses in the other diagonals are found in precisely the same manner as for Cd. For the inclined end post aB the lines of action of R and r are coincident, so that this stress may be most readily found by multiplying the shear, $R-r$, by 1.38, the stress in aB for $R=1$.

The stress in the verticals from the loads r at their lower extremities is $+r$, which must be subtracted from the compressive stress due to R, as in Eq. (30), Art. 136.

The resultant web stresses, S, thus obtained, as well as their exact values as found by computation in the preceding article, are as follows:

	S	
	Graphic.	Exact.
$aB = (342.5-25.2 \;=317.3)\times\ 1.38 =$	(−)438	(−)438.2
$Bc = (239.0\times 1.05 \;=251\;\;)-15 \;=$	(+)236	(+)237.5
$Cd = (164.2\times 1.09 \;=179\;\;)-15 \;=$	(+)164	(+)163.8
$De = (101.3\times 1.28 \;=130\;\;)-16 \;=$	(+)114	(+)113.9
$Ef = (\;52.0\times 1.65 \;=\;\;86\;\;)-16 \;=$	(+) 70	(+) 69.5
$Gf = (\;15.8\times 2.195 = \;\;35\;\;)-\;6 \;=$	(−) 29	(−) 28.8
$Cc = (164.2\times 0.765 = 126\;\;)-12.1 =$	(−)114	(−)113.5
$Dd = (101.3\times 0.835 = \;\;84\;\;)-12.1 =$	(−) 72	(−) 72.3

The live-load stress in the counter *Ef* should be diminished by 18.7, as explained in the example in Art. 134, on account of the dead-load tensile stress in *Fe*. This value is one-half the dead-load compressive stress in *Ef*, if *Fe* did not exist (Art. 102), and that stress might have been found graphically in connection with Fig. 8, Art. 133.

The agreement between the values of S in the last two columns is seen to be very satisfactory. Comparing the graphic method, as here applied to the stresses from a concentrated load system, with the numeric method, it is seen that the former is dependent on known values of R and r, and is applicable to web stresses only. With the aid of a slide rule the stresses may be obtained more quickly as well as more accurately than by the graphic method.

138. Live-load Stresses from Uniform Loading by the Exact Method. The stresses in trusses with inclined chords are rarely computed from uniform loading by the exact method. For highway bridges the conventional method, Art. 134, is usually employed, since uncertainties as to the actual loading and its distribution under service conditions do not warrant greater refinement in the assumptions. For railway bridges the stresses are commonly based on concentrated load systems.

(*a*) *Main Diagonals.* Referring to Fig. 16, consider the stress in main diagonal *Cd*. The criterion for concentrated loads, Eq. (25), Art. 136, is perfectly general, since a uniform load may be regarded as a special case of concentrated loading (Art. 38). Hence

$$W=\frac{u}{v}\frac{l}{p}W_1$$

and assuming the intensity of the uniform load per truss as unity,

$$a+x=\frac{u}{v}\frac{l}{p}x;$$

or

$$x=\frac{av}{ul-vp}p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

A load in panel *cd* within the distance x produces tension in *Cd*, or compression in *Dc* if *Cd* does not exist, and a load beyond that distance produces compression in *Cd* or tension in *Dc*; hence the stress in both members from a load distant x from d is zero. This affords another means of determining x. Assuming a load, unity, at the point referred to, the resultant moment of the forces to the left of section 1 about G equals zero. Hence

$$\frac{a+x}{l}v-\frac{x}{p}u=0,$$

from which the value of x in Eq. (35) may be derived. If the truss has N equal panels, and if n denotes the number of panels in the distance a,

$$x=\frac{nv}{Nu-v}p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

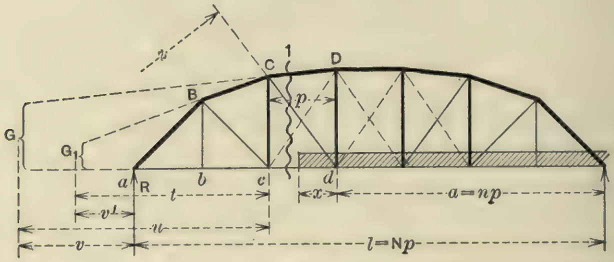


FIG. 16.

in which v and u may conveniently be expressed in panel lengths and p and x in feet.

After computing x by Eq. (36) the left end reaction R, and the forward apex load, r, at c may be obtained by the following equations, w denoting the intensity of the uniform load per truss:

$$R=\tfrac{1}{2}w(a+x)^2/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and

$$r=\tfrac{1}{2}wx^2/p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In these equations $w/2l$ and $w/2p$ are constants, so that all the values of R may be obtained by a single setting of the slide rule, and all the values of r by another setting.

The stress S_d is then readily found by Eq. (27), Art. 136. Eq. (35) may be expressed,

$$x=\frac{a}{\dfrac{u}{v}l-p}p.$$

For parallel chords, v=infinity. As v approaches infinity, u/v approaches unity. Therefore, when both chords are horizontal, $x=ap/(l-p)$ which agrees with Eq. (14), Art. 98, as it should.

(*b*) *Verticals.* For maximum stress in the vertical post Cc, the loading must extend a distance into the panel which may be found by Eq. (35) or (36) after v' and t, the distances to the new moment center G_1, are substituted for v and u respectively. Thus

$$x=\frac{av'}{tl-v'p}p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

or, if the panels are equal,

$$x=\frac{nv'}{Nt-v'}p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

If the slope of BC is the same as that of CD, then G_1 and G coincide, and x has the same value for members Cd and Cc meeting in the unloaded chord at C.

After x has been determined by Eq. (40), R and r may be found by Eqs. (37) and (38) and the corresponding stress in the post by Eq. (30), Art. 136.

The maximum tensile counter stress in the vertical posts will be considered later (Art. 141).

(*c*) *Counters.* Referring to Fig. 17, consider the stress in counter Dc, assuming at first that the main diagonal Cd in the same panel is omitted. In the analysis for concentrated loading, Art. 136, it was seen that the criterion for the counters, Eq. (33), is the same as that for the main diagonals, Eq. (25). That relation applies also to the present case. Hence with due regard to the notation in Fig. 17, the distance x may be found by Eq. (36), and R and r by Eqs. (37) and (38). The corresponding stress in the counter is then

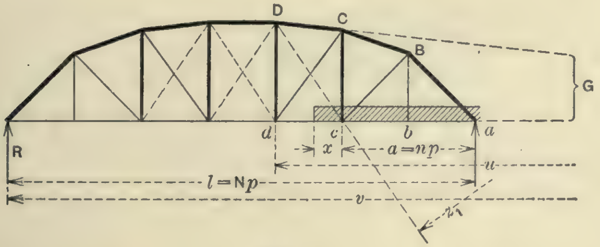


FIG. 17.

$$S_c=\frac{Rv-ru}{z_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

As explained in Art. 134 (*c*), this should be reduced by $\frac{1}{2}S_d\, i_1/i$, for the dead-load tensile stress S_d in *Cd* (or the dead-load compressive stress $S_d\, i_1/i$ in *Dc* when *Cd* is not in action) in which i_1 and i are the lengths of *Dc* and *Cd* respectively.

A load in panel *cd* distant x from c will produce no stress in either diagonal in that panel. A load anywhere between the left support and the head of the uniform load will produce tension in *Cd*; hence for maximum tension in *Cd* that entire distancemust be covered. Thus it is seen also that *the sum of the distances which the load must advance into a given panel, from opposite sides, for maximum stress in the main diagonal and counter, must equal the panel length.* This affords a convenient check on the numeric values of x (see example following).

(*d*) *Chords.* All chord stresses become maximum when the uniform load covers the entire bridge, and the stresses may be found as explained in Art. 129.

Example. Required the maximum live-load stresses in the trusses of the 200-ft. single-track railroad bridge shown in Fig. 18, from a uniform load of 7400 lbs. per foot of bridge, by the exact method.

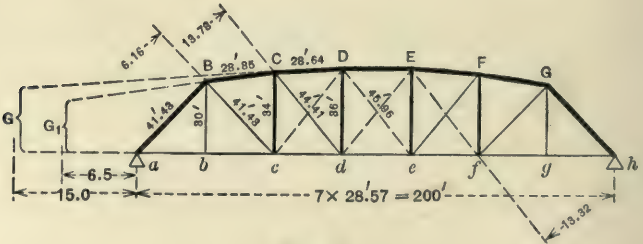


FIG. 18.

Solution. All loads and stresses will be expressed in thousands of pounds. The values of the lever arms shown in the figure were computed in the example in Art. 132.

For maximum stress in the inclined end post and in all the chord members, the uniform load must extend over the entire span. These stresses are, therefore, the same as those previously found in Art. 134, in the example illustrating the conventional method of treating uniform loading.

Diagonals. The distances, x, from the right ends of the panels containing the diagonals, which must be covered by the uniform load may be found by Eq. (36) for *Bc*, *Cd*, *Ef* and for the counter stress in *Gf*, and by Eq. (15), Art. 98, for *De*, since the chords in that panel are both horizontal. These distances are as follows:

$$\text{For } Bc,\ x=\frac{5\times 6.5}{7(6.5+1)-6.5}\times 28.57=20.2 \text{ ft.}$$

$$\text{For } Cd,\ x=\frac{4\times 15}{7(15+2)-15}\times 28.57=16.5 \text{ ft.}$$

$$\text{For } De,\ x=\frac{3}{7-1}\times 28.57=14.3 \text{ ft.}$$

$$\text{For } Ef,\ x=\frac{2(7+15)}{7(15+3)-(7+15)}\times 28.57=12.1 \text{ ft.}$$

$$\text{For } Gf,\ x=\frac{1(7+6.5)}{7(6.5+2)-(7+6.5)}\times 28.57=8.4 \text{ ft.}$$

As a check, note that $(20.2+8.4)=(16.5+12.1)=(14.3\times 2)=28.6$ ft., the panel length.

The values of R and r may now be found by Eqs. (37) and (38), in which $w=7.4/2=3.7$.

These values are:

	Bc	*Cd*	*De*	*Ef*	*Gf*
$R=$	246.1	158.3	92.5	44.3	12.7
$r=$	26.4	17.6	13.2	9.5	4.6

The stresses in all the diagonals except *De* may then be found by Eq. (27), Art. 136. Stress *De* is the product of the shear in the panel and the secant of the angle that member makes with a vertical. Thus,

$$Bc=(246.1\times 6.5-26.4\times 7.5)\div 6.16=(+)227.6;$$

$$Cd=(158.3\times 15.0-17.6\times 17)\div 13.78=(+)150.6;$$

$$De=(92.5-13.2)\frac{45.96}{36}=(+)101.2;$$

$$Ef=(44.3\times 22-9.5\times 18)\div 13.32=(+)60.4;$$

$$Gf=(12.7\times 13.5-4.6\times 8.5)\div 6.16=(-)21.5.$$

The live-load stress in the counter *Ef* should be diminished by 18.7, as explained in the example in Art. 134, on account of the dead-load tensile stress in *Fe*; hence,

$$Ef=+60.4-18.7=(+)41.7.$$

The dead-load stress in *Gf* was found to be +80.7 in the example, Art. 132. The minimum stress in this member is therefore $+80.7-21.5=(+)59.2$.

The minimum stress in the counters and in the main diagonals in panels containing counters is zero.

Vertical Posts. By Eq. (40):

$$\text{For } Cc,\ x=\frac{4\times 6.5}{7(6.5+2)-6.5}\times 28.57=14.0 \text{ ft.}$$

$$\text{For } Dd,\ x=\frac{3\times 15}{7(15+3)-15}\times 28.57=11.6 \text{ ft.}$$

The corresponding values of R and r by Eqs. (37) and (38) are:

$$\text{For } Cc,\ R=152.3 \text{ and } r=12.7;$$

$$\text{For } Dd,\ R=\ 87.6 \text{ and } r=8.7.$$

The stresses by Eq. (30) Art. 136, are then,

$$Cc=\left(152.3\times\frac{6.5}{8.5}\right)-12.7=(-)103.8;$$

$$Dd=\left(\ 87.6\times\frac{15}{18}\right)-\ 8.7=(-)64.3.$$

The maximum live-load tensile stresses in the posts will be considered later (Art. 141).

The maximum live-load stress in the hip vertical is,

$$Bb=\frac{7.4}{2}\times 28.57=(+)105.7.$$

139. Comparison of Maximum Live-load Stresses Found by Different Methods.

The live-load stresses in the trusses of the 200-ft. single-track railway bridge, shown in Fig. 18, as computed in the examples in preceding articles, are tabulated below for convenient comparison. The dead-load stresses, Art. 132, and the stresses by graphic methods, Arts. 133, 135 and 137, are not included. All stresses are expressed in thousands of pounds, and the signs indicating their character are omitted.

The table comprises:

I. The stresses from Cooper's E–60 loading, Art. 136.

II. The stresses from an 'equivalent' uniform load of 7400 lbs. per foot of bridge, which gives the same maximum shear in the end panel as Cooper's E–60 loading (Art. 137) by

   (*a*) The conventional method, Art. 134.

   (*b*) The exact method, Art. 138.

Web Members.				Chord Members.			
Member.	Loading.			Member.	Loading.		
	I	II			I	II	
		(*a*)	(*b*)			(*a*)	(*b*)
aB	438.2	437.8	437.8	*ab, bc*	302.2	302.1	Same as (*a*)
Bc	237.5	239.0	227.6	*cd*	430.0	444.0	
Cd	163.8	164.4	150.6	*de*	476.4	503.2	
De	113.9	115.7	101.2				
Ef	50.8	56.1	41.7	*BC*	434.2	448.4	
Gf	28.8	33.1	21.5	*CD*	489.3	504.5	
				DE	488.1	503.2	
Bb	125.2	105.7	105.7				
Cc	113.5	115.4	103.8				
Dd	72.3	75.5	64.3				

From this table it is seen:

First. That the stresses (II *a*) computed by the *conventional* method from an ‘equivalent’ uniform loading giving the same shear in the end panel as the concentrated load system, are either substantially equal to the stresses (I) from the latter, or that the differences are on the side of safety. The only exception is the stress in the hip vertical, *Bb*, which should always be obtained by computing the maximum floor-beam reaction from the loading governing the design of the floor system.

Second. That the web stresses (II *b*) computed by the exact method from the ‘equivalent’ uniform loading derived as above stated, involve differences on the side of danger compared with the stresses (I) from the concentrated load system, and that the percentage of difference increases rapidly as the magnitude of the stresses decreases.

The relations in the above table between the stresses found by the three methods under comparison are not peculiar to the data chosen for the particular truss to which they apply, but they are fairly indicative, in a general way, of the comparative results to be expected by these methods. Accepting the stresses (I) from the concentrated load system as the logical basis of comparison, it may be concluded:

I. That the computation of the stresses from an ‘equivalent’ uniform loading giving the same shear in the end panel as the concentrated load system is admissible with slight errors on the side of safety, provided

(*a*) the stresses are computed by the *conventional* method, and

(*b*) the stress in the hip vertical is based on the loading governing the design of the floor system.

II. That the computation of the stresses from the ‘ equivalent ’ uniform loading described, by the *exact* method, is inadmissible if the bridge is to be designed for the stresses from a concentrated load system with no errors on the side of danger.

140. Web Stresses by Vertical Intercepts Instead of Lever Arms. (*a*) *Diagonals.* Referring to Fig. 19, it has been shown, Art. 136, that for maximum stress in *Cd* the critical load is at *d*, and that if *r* denotes the forward apex load at *c*, the stress in the diagonal is

$$S_d=\frac{Rv-ru}{z}.$$

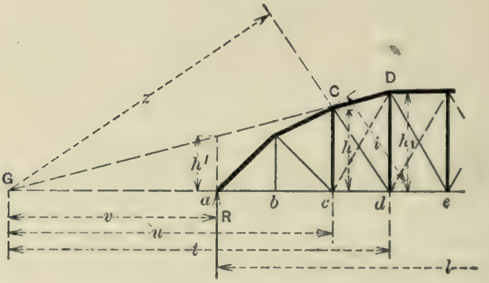


FIG. 19.

From similar triangles, $z=th/i$, whence

$$S_d=R\frac{v}{t}\,\frac{i}{h}-r\frac{u}{t}\,\frac{i}{h}.$$

Again, by similar triangles, $v/t=h'/h$, and $u/t=h/h_1$. Hence

$$S_d=\left(R\frac{h'}{h}-r\right)\frac{i}{h_1}. \qquad (42)$$

In this equation h, h_1 and i are known lengths of truss members and h' is the vertical intercept from the end support to the inclined chord member prolonged. In applying Eq. (42) it is convenient to express all dimensions in feet. If both chords are horizontal, $h' = h = h_1$ and $S_d = (R - r)i/h$, which represents the product of the shear in the panel and the secant of the angle between the diagonal and a vertical, as it should.

If the stress is computed by the conventional method for uniform loading, Art. 134, $r = 0$ and Eq. (42) becomes

$$S_d = R\frac{i}{h}\frac{h'}{h_1}, \quad \ldots \ldots \quad (43)$$

which differs from the value of S_d for both chords horizontal by the added factor h'/h_1.

(*b*) *Verticals.* The moment center of post *Dd* lies at *G*, and for maximum stress in this member the critical wheel is placed at *e*. Letting *r* denote now the forward apex load at *d*, the stress in the post is

$$S_p = R\frac{v}{t} - r,$$

or

$$S_p = R\frac{h'}{h_1} - r. \quad \ldots \ldots \quad (44)$$

If both chords are horizontal $h' = h_1$ and $S_p = R - r =$ the shear in panel *de*.

If the stress is computed by the conventional method for uniform loading, $r = 0$ and Eq. (44) becomes

$$S_p = R\frac{h'}{h_1}. \quad \ldots \ldots \quad (45)$$

Eqs. (42) and (44) may be expressed as follows, c and c_1 being constants:

$$S = cR - c_1 r. \quad \ldots \ldots \quad (46)$$

For the diagonals, $c = h'/h_1 \times i/h$ and $c_1 = i/h$, in which h and h_1 are the heights of the truss at the ends of the panel containing the diagonal, and $h_1 > h$.

For the posts, $c = h'/h_1$ and $c_1 = 1$, in which h_1 is the length of the post.

The values R and r may be obtained with the aid of a moment table, for the load in position for maximum stress in each web member, and the stresses may then be computed by the general Eq. (46).

From Art. 136 it is seen that the live-load stresses in the counters may be obtained in the manner above explained for the main diagonals, except that h' denotes then the intercept at the *farther* end of the truss from the counter, and h_1 is then the *lower* height of the truss in the panel containing the counter.

In conclusion it may be said that there is little ground for choice between the formulas in this article and those in Art. 136. The numeric values of the constants c and c_1 in Eqs. (31) and (46) must, of course, be the same in any given case. Algebraically the constants in Eq. (46) are expressed, however, in terms of the known dimensions of the truss and the vertical intercept, h', at the end support which has to be computed for each inclined chord member.

141. Tension in Vertical Posts. In previous articles attention has been called to the fact that under certain conditions the vertical posts in a Pratt truss with inclined chords are in tension. These tensile stresses are relatively very small compared with the compressive stresses. Since such members are, however, subject to a sudden reversal of stress during the passage of a single train, the working stresses governing their design are usually much lower than those for posts subject only to compression. Again, the reversal of stress in members with 'pin-connected' ends causes a sudden change of pressure from one side of the pin to the other, which tends eventually to cause permanent deformation of the pin holes, thus increasing the play between the pin and the pin hole, unless the allowable working stress in bearing is suitably reduced to prevent this effect. For these reasons, the possibility of reversal of stress in the posts requires careful investigation, although the tensile stresses themselves are quite small. The analysis of these stresses and the position of loading for which they become maxmium will now be considered.

Uniform Load. Referring to Fig. 20, let it be required to find the conditions for maximum tensile stress in post Cc from a uniform load of intensity w per truss. Let D denote the dead-load tensile stress in diagonal Cd, W' the dead-load concentration at C, and let the loading be assumed to advance from the right. It is to be carefully noted that in all that follows, R and R_1 denote the reactions from the live load under consideration, exclusive of the dead load. The counter Dc is to be considered so adjusted that it will not come into action until the tensile stress, D, in Cd is neutralized by the compressive stress from the live-load.* Assume that condition to be met when the head of the live load is at k_1. Then $Rv = \frac{1}{2}Dz$ (Art. 102), and the stresses in Cd and Dc are momentarily zero. Analyzing then the forces at C, it is seen from $\Sigma H = 0$ that the horizontal components of the resultant chord stresses DC and CB from dead and live loads combined must be equal. Let S_h denote this component, h_1 and h_2 the vertical projections of CB and DC, respectively, and p the panel length. Then the tensile stress in Cc is

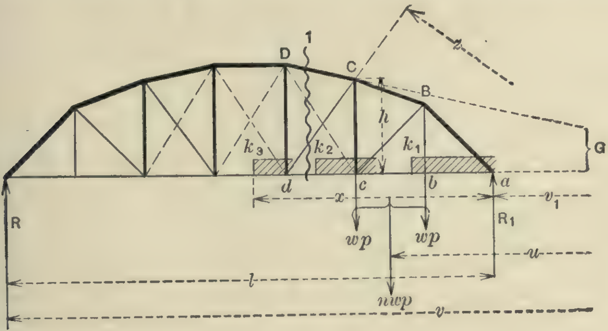


FIG. 20.

$$S_t = \frac{S_h}{p}(h_1 - h_2) - W', \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

in which S_h is found by dividing the total bending moment at c from the dead and live loads by the height h.

As the load advances further to the left, the counter Dc is brought into action, its stress increasing till the head of the load reaches some point k_2 in the panel cd when the stress becomes maximum. As the load advances beyond k_2 the stress in the counter decreases progressively till, when the head of the load reaches some point k_3, this stress becomes zero. The stress in the other diagonal Cd has remained at zero as the load advanced from k_1 to k_3. During this movement the bending moment at c and therefore S_h and S_t in Eq. (47) have continually increased. As the load advances beyond k_3, the diagonal Cd is again brought into action—that is to say, any additional load to the left of k_3 induces tension in Cd, and compression in Cc. The tension in Cc is thus seen to become maximum when the head of the load is at k_3. The position of k_3 depends on the relation of the live to the dead load and on the proportions of the truss. As a rule k_3 will lie to the left of the panel cd containing the diagonal Cd which meets the post in the unloaded chord, but it may lie within that panel. The problem must be analyzed for each of these conditions.

---

* This assumption is made merely to simplify the analysis. Practically the counters are usually under some initial tension, but unless this is very excessive through improper adjustment, it has either no effect on the resultant maximum tension, or a slight and negligible effect, as has been shown in Arts. 74 and 81, and it may therefore be disregarded in this analysis.

(*a*) *Head of Load to Left of Panel.* Since the stresses Cd and Dc, Fig. 20, are zero for the position k_3, that position must be such as to produce a live-load compressive stress of $\frac{1}{2}D$ in Cd, for such a stress, computed on the basis of a static live load, is assumed to neutralize momentarily, when the load is in motion, a dead-load tensile stress of D. Considering the live loads to the right of section 1, the above condition is met when the load extends to such a point k_3 that R_1 will satisfy the following equation:

$$\frac{nwp \times u - R_1 v_1}{z} = \tfrac{1}{2}D,$$

whence

$$R_1 = \frac{nwp \times u - \frac{1}{2}Dz}{v_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

in which the lever arms u, v_1 and z may be conveniently expressed in panel lengths. In deriving Eq. (48) a full panel load wp was assumed at c, which implies that the loaded length x is at least equal to the distance from a to d. To ascertain whether this implied condition is met in any given case find the value of R_1 corresponding to a load from a to d. If this value is greater than R_1 from Eq. (48), it shows that k_3 must lie *within* the panel cd to satisfy that equation, and that Eq. (48) is therefore not applicable. If, on the other hand, this value is less, the conditions assumed in the derivation of Eq. (48) are met. The corresponding live-load tensile stress, S_l, in post *Cc* may then be obtained from the moment of the forces to the right of section 1, Fig. 21, about G_1, the intersection of *CB* and *dc*, viz.:

$$S_l = \frac{(nwp \times u') - R_1 v_1'}{t}. \quad \ldots \ldots \quad (49)$$

In this equation R_1 has the value found by Eq. (48), and $u,'$ t and v_1' may be conveniently expressed in panel lengths.

The diagonals *Cd* and *Dc* are omitted in Fig. 21, since the stresses in these members are zero. Under dead loading alone *Cd* is in action, so that section 1 may be taken without severing more than three members, and the moment center of *Cc* remains at G_1. The dead-load compressive stress S_d in *Cc* may therefore be directly combined with S_l, after multiplying the former by one-half, for the same reason as for the dead-load stress in diagonal *Cd*. Thus,

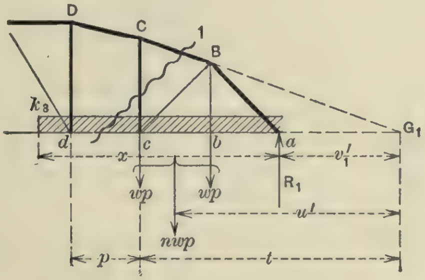


FIG. 21.

$$S_t = S_l - \tfrac{1}{2}S_d \quad \ldots \ldots \quad (50)$$

(*b*) *Head of Load within Panel.* The conditions in this case are as shown in Fig. 22. It is more convenient now to consider the moment about *G* of the forces to the left of section 1, Letting *r* denote the forward apex load at *d*, then for zero stress in *Cd*,

$$\frac{Rv - ru}{z} = \tfrac{1}{2}D$$

and

$$R=\frac{ru+\frac{1}{2}Dz}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Taking section 2, and moments about G_1:

$$S_l=\frac{Rv'-ru'}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and S_t may then be obtained, as before, from Eq. (50).

For the solution of Eqs. (51) and (52) r must be known and therefore x. Letting $r=\frac{1}{2}wx^2/p$ and $R=\frac{1}{2}w(a+x)^2/l$, Eq. (51) becomes

$$\frac{vw(a+x)^2}{2l}=\frac{uwx^2}{2p}+\tfrac{1}{2}Dz.$$

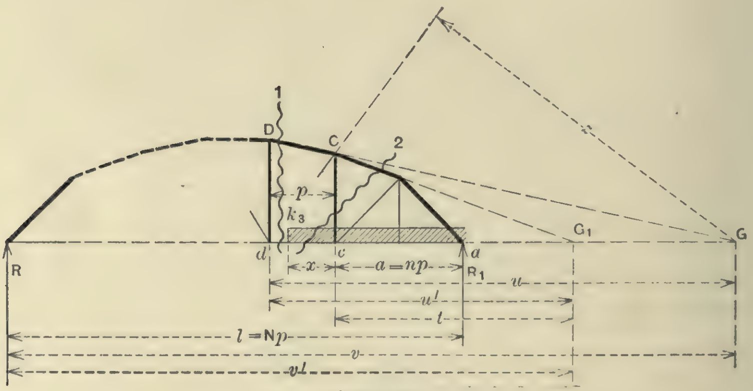


FIG. 22.

Letting $l=Np$, $a=np$, and $D/wp=c$.

$$v(np+x)^2-Nux^2=Ncp^2z.$$

Expressing u, v, x and z in panel lengths, and canceling p^3,

$$v(n+x)^2-Nux^2=Ncz,$$

or

$$(Nu-v)x^2-2nvx=n^2v-Ncz \quad . \quad . \quad . \quad . \quad (53)$$

This equation is applicable only if all lengths are expressed in panel lengths.

By assigning suitable values in any given case to the constants in Eq. (53) the value of x (in terms of the panel length) is found by solving the resulting quadratic. Then, after converting x into feet, the values r and R in Eqs. (51) and (52) are readily found from the equations:

$$r=\tfrac{1}{2}wx^2/p \quad \ldots \ldots \ldots \quad (54)$$

and

$$R=\tfrac{1}{2}w(a+x)^2/l, \quad \ldots \ldots \ldots \quad (55)$$

in which all dimensions are expressed in feet.

Concentrated Loads. For a concentrated live-load system the position corresponding to k_3 must be found by trial. As the moment table is constructed it is more convenient then to consider the moment M_G about G, Fig. 20, of the live loads to the left of the section, whence

$$Rv-M_G=\tfrac{1}{2}Dz, \quad \ldots \ldots \ldots \quad (56)$$

in which v, z and the lever arms entering M_G may be conveniently expressed in panel lengths.

For maximum tension in post Cc, the loading must be so placed as to satisfy Eq. (56), in which the second member is a constant. For the first trial position, place that wheel at c which will cause the load to advance as far into panel cd as possible without bringing the first wheel to the left of d, and then shift the load as may be required to satisfy Eq. (56). After the position for maximum tension in the post has been thus determined S_l may be best found by Eq. (52), in which M_{G_1} the moment of the live loads to the left of the section about G_1, Fig. 22, should be substituted for ru', in order to make the equation applicable also in case the loading should extend to the left of d. This equation then becomes:

$$S_l=\frac{Rv'-M_{G_1}}{t}. \quad \ldots \ldots \ldots \quad (57)$$

Finally, S_t may be found, as before, from Eq. (50).

The vertical post, Dd, which meets the upper chord at the point where its inclination from the horizontal begins, is in tension from the dead load, as seen in the example, Art. 132. The live-load tension in this post is maximum when chord stress CD is maximum, for neither of the counters at D is then in action, as was seen in Art. 119, and in the example, Art. 136.

The tensile stress in *Dd* is therefore the vertical component of the dead and live-load compressive stress in *CD*.

Example. Required the maximum tensile stresses in the vertical posts in the 200-ft. single-track railroad bridge shown in Fig. 23, for (*a*) Cooper's E–60 loading, (*b*) a uniform load of 8000 lbs. per foot of bridge and (*c*) a uniform load of 3000 lbs. per foot of bridge.

Solution. All stresses and loads will be expressed in thousands of pounds.

From the example in Art. 132, the dead-load stresses needed are as follows:

$$Cd = +36.1 \quad Cc = -30.4 \quad Dd = +0.4.$$

(*a*) *From Cooper's E–60 Loading:*

Post Cc. From Eq. (56), the load must be so placed that $Rv - M_G$ equals $\frac{1}{2}Dz$, which has the value $\frac{1}{2} \times 36.1 \times 13.78 = 249$.

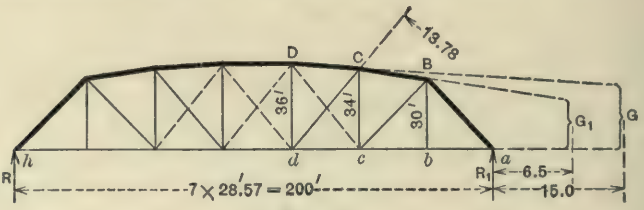


FIG. 23.

For the first trial position place wheel 5 at *c*. Wheel 1 is then 5.6 ft. to the right of *d*. The reactions at *d* and *h*, found with the aid of the moment table, are as follows:

$$r = 43.6 \quad \text{and} \quad R = 67.4,$$

whence

$$Rv - M_G = 67.4(15+7) - 43.6(15+3) = 698.$$

Since this is not equal to 249, the condition for maximum tension in *Cc* is not fulfilled.

For the second trial position place wheel 6 at *c*. Wheel 1 is then 3.4 ft. (=0.12 panel length) to the left of *d*. Letting *r* denote, as before, the reaction at *d* from the loads in panel *cd*, $r = 69.3$ and $R = 83.1$. In this case

$$Rv - M_G = 83.1 \times 22 - (69.3 \times 18 + 15 \times 18.12) = 309,$$

which still is greater than 249.

Advancing the load 1 ft. to the left, $r = 74.2$ and $R = 85.0$, whence

$$Rv - M_G = 85 \times 22 - (74.2 \times 18 + 15 \times 18.15) = 262,$$

which is sufficiently close to 249 for practical purposes.

With the loading in this position, S_l may be found by Eq. (57) in which

$$M_{G_1}=74.2(6.5+3)+15(6.5+3.15)=850,$$

whence

$$S_l=\frac{85(6.5+7)-850}{6.5+2}=(+)\ 35.0;$$

and finally by Eq. (50),

$$S_t=+35.0-\frac{30.4}{2}=(+)19.8.$$

Post Dd. The greatest tensile stress in *Dd* is equal to the vertical component of the greatest compressive stress in *CD*. The latter, from the example in Art. 136, is 489.3, whence

$$Dd=489.3\times\frac{2}{28.64}=(+)34.2.$$

As previously found in the Example in Art. 132, this post is in tension from the dead load, this stress being +0.4.

(*b*) *From a Uniform Load of* 8000 *lbs. per foot of Bridge.* The panel load, per truss, is 28.57×4=114.3.

Post Cc. By Eq. (48),

$$R_1=\frac{(2\times114.3\times16.5)-(\frac{1}{2}\times36.1\times13.78)}{15.0}=234.9.$$

The value of R_1 for the load extending from *a* to *d* is $114.3(2\times5.5+\frac{1}{2}\times4)/7=212.3$. Since $212.3<234.9$ the head of the load is to the left of *d*, and Eq. (49) is therefore applicable, whence

$$S_l=\frac{(2\times114.3\times8)-(234.9\times6.5)}{8.5}=(+)35.5,$$

and by Eq. (50),

$$S_t=35.5-\frac{30.4}{2}=(+)20.3.$$

Since the head of the load is to the left of *d* its exact position is not needed for the immediate purpose in view. As a matter of interest it may be stated, however, that for $R_1=234.9$ the head of the load is 10.4 ft. to the left of *d*. If the live-load is reduced from 8000 to 6000 lbs. per foot the head of the load would lie 6.7 ft. to the left of *d* for maximum tension in *Cc*, and if the live load is 4000 lbs. per foot, this distance becomes 2.6 ft. This shows that for a given truss the position of loading for maximum tension in a post is only slightly affected by the intensity of the loading.

Post Dd. The maximum live-load stress in *CD* is −545.5. Hence, as explained above,

$$Dd=545.5\times\frac{2}{28.64}=(+)38.1.$$

(*c*) *From a Uniform Load of* 3000 *lbs. per foot of Bridge.*

Post Cc. For this loading, by Eq. (48) $R_1=77.7$ and for the load extending from a to d, $R_1=79.6$. Since $79.6>77.7$, the head of the load is to the right of d, and Eq. (52) is therefore applicable, but it is first necessary to find the value of x.

From Eq. (53)

$$(\overline{7\times18}-22)x^2-(2\times2\times22)x=\left(4\times22\right)-7\times\frac{36.1}{1.5\times28.57}\times13.78,$$

whence

$$104x^2-88x=+6.74,$$

and

$$x=0.917 \text{ panel length},$$

or

$$x=0.917\times28.57=26.2 \text{ ft.}$$

Then from Eqs. (54) and (55), $r=18.0$ and $R=26.0$, and from Eq. (52),

$$S_l=\frac{26(6.5+7)-18.0(6.5+3)}{6.5+2}=(+)21.2.$$

Finally, from Eq. (50),

$$S_t=21.2-\frac{30.4}{2}=(+)6.0.$$

142. Deck Truss with Reduced Height at Ends.

In deck trusses the lower chord in the two end panels is sometimes given an upward inclination toward the ends as shown in Fig. 24 (*b*).

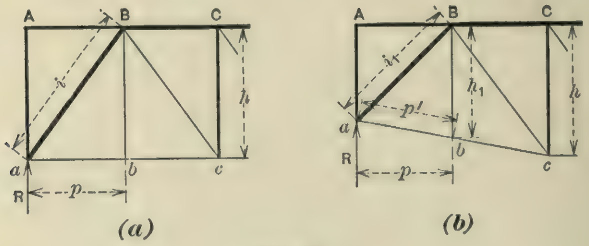


FIG. 24.

A saving of metal is thus effected in comparison with a truss of constant depth, Fig. (*a*), if the conditions are alike in other respects. This saving is usually more than neutralized, however, by the cost of the additional masonry required to carry the bridge seats at a to a higher level.

An analysis of the stresses in Figs. (*a*) and (*b*) will show the following comparison:

It is seen in both figures from joint A that stress Aa is equal to the load at A, and that stress AB is zero; also that stress Bb is equal to the load at b. Again, in both cases stress BC is equal to the bending moment at c divided by h. The only stresses affected by the inclination of the lower chord are aB, Bc and $ab=(bc)$.

Since stress Aa does not affect the stress in any other member, let R denote the end reaction, after subtracting from the actual reaction an amount equal to stress Aa. Member Aa may then be assumed omitted in the analysis.

In Figs. (a) and (b) the triangle aBb may be regarded as a force triangle of the three forces, R, stress aB and stress ab, meeting at a. The scale of forces is then $R \div \overline{Bb}$, or R/h in Fig. (a) and R/h_1 in Fig. (b). Hence,

In Fig. (a) In Fig. (b)

$$\text{Stress } aB = \frac{R}{h}i; \qquad \text{Stress } aB = \frac{R}{h_1}i_1;$$

$$\text{Stress } ab = \frac{R}{h}p; \qquad \text{Stress } ab = \frac{R}{h_1}p'.$$

From considerations of appearance as well as economy the panel length, p, is usually made less than the height of truss, h. Again, on grounds of appearance (not economy, in so far as the truss, apart from the masonry, is concerned) it is usually considered undesirable that the angle between the inclined end post aB, Fig. (b), and a horizontal line should be less than 45°. Under these restrictions the ratio i/h is very nearly equal to, and usually slightly greater than i_1/h_1, so that the stress in the inclined post, aB, in Fig. (a) is usually somewhat greater than in Fig. (b). Again, from geometric relations, $p/h < p'/h_1$. Hence the stress in the end lower chords in Fig. (a) is less than in Fig. (b). That may be seen also by observing the difference in the lengths of the lever arms in Figs. (a) and (b) of stress ab about B.

Stress Bc is greater in Fig. (a) than in Fig. (b), since in the latter the shear in panel bc is partly resisted by the vertical component of stress bc. The maximum stress in Bc, Fig. (b), may be found as explained in previous articles on the analysis of trusses with inclined upper chords.

The net saving in Fig. (b) is due essentially to the reduction in length of members *Aa*, *Bb* and *aB*, the saving of material in *Bc* being usually about offset by the increase of material in *ab* and *bc*. The reduction in length of the compression members *Aa* and *aB* may also render a slight increase of unit working stresses in these members permissible, although the saving from that source is very slight.

143. Triangular Truss with Inclined Chords. The stresses in the diagonal web members of a triangular truss with inclined upper chord, as in Fig. 2, Art. 129, or inclined lower chord, as in Figs. 25 (*a*) and (*b*), may be found in precisely the same manner as explained for the Pratt truss with inclined chords in preceding articles. The stress in any vertical is equal to the load at the joint at which that member alone meets the chord.

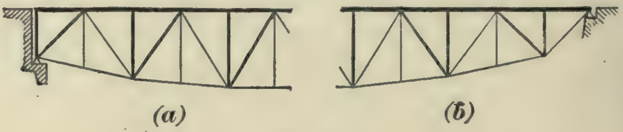

(*a*) (*b*)

FIG. 25.

144. Parabolic Truss. A parabolic truss, Fig. 26, is one in which the joints of one chord lie in parabola traversing the end supports. If the web system is of the Pratt type the horizontal components, S_h, of the stresses in the inclined chord members for a uniform load covering the entire bridge are constant; for

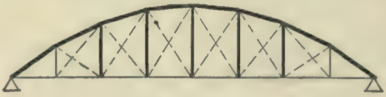


FIG. 26.

in the equation $S_h = M/h$, in which M is the bending moment at a joint at which the depth of truss is h, both M and h will then vary as the ordinates to parabolas having the span-length as a common chord, so that S_h is constant. It follows, therefore, from $\Sigma H = 0$ applied to the forces at any joint in the inclined chord, that *under uniform loading covering the whole bridge the stress in every diagonal is zero, and the stress throughout the horizontal chord is constant.* The value, S_h, of this chord stress may be found by dividing the bending moment at any joint in the inclined chord by the height of the truss at that joint. All the diagonals may therefore be regarded as counters, and if they are to act only in tension under partial loading it is necessary to provide a pair of diagonals in every panel. If w denotes the intensity of the uniform load per truss, l the span length, and p the panel length, then for an even number of panels, $M_{\max}=\frac{1}{8}wl^2$ and for an odd number of panels, $M_{\max}=\frac{1}{8}w(l^2-p^2)$, Art. 42, Eqs. (22) and (23). Accordingly, $S_h=M_{\max}/h_{\max}$, and the stress in any member of the inclined chord, of length p', is $S_h \times p'/p$. Again, for a truss of N equal panels, the reaction $R=\frac{1}{2}(N-1)wp$, and if h_1 is the height of the truss at the hip joint, $S_h=R\times p/h_1=\frac{1}{2}(N-1)wp^2/h_1$.

When the uniform load extends over the entire bridge, the tensile stress in every vertical is wp, and this is the maximum tension in these members. The maximum compressive stresses in the verticals and tensile stresses in the diagonals may be found by the methods explained in previous articles. The former are constant for all heights of truss (Art. 131), for by the law of the parabola any change of height will affect all verticals in like ratio. If a load wp is applied directly at every joint of the parabolic chord, the stresses in the verticals from such loading are zero. The stress in any diagonal will be affected by a change in the height of the truss in proportion to the change in the secant of the angle which it makes with the vertical (Art. 131).

In general, for a uniform load covering the entire bridge the figure formed by the chords may be regarded as an equilibrium polygon for that loading, the pole distance having the value $M_{\max}/h_{\max}$. The bending moment at any section is then the product of the pole distance and the vertical intercept between the chords. The use of the parabolic truss is restricted, in general, to short highway bridges of the pony type, for which the stresses are usually computed by the conventional method for uniform loading.

The curve traversing the end supports and the joints of one of the chords may be a segment of a circle or any other regular curve. The general term *bowstring truss* is applied to trusses of this type.

145. Double-parabolic Truss. In a double-parabolic or 'lenticular' truss, Fig. 27, the joints in both chords lie in parabolas. Hence the vertical intercepts between the chords vary as the ordinates to a parabola. The conditions referred to in the preceding article are therefore equally applicable to the double-parabolic truss in so far as the stresses in the chords and diagonals are concerned.

The stresses in the verticals from a uniform load covering the entire bridge—as the dead load, for example—may be either tensile or compressive. Considering the general case of two parabolic chords of unequal curvature, if a section cutting both chords is taken anywhere, it follows from $\Sigma H=0$ applied to the forces on either side of the section, that S_h, the horizontal component of the chord stress, has the same value for both chords, and, therefore, since such a section may be taken anywhere, S_h is constant throughout both chords. Analyzing the forces meeting at a lower-chord joint, let p denote the horizontal panel length, h and h_1 the vertical projections of the chord members to either side of the joint, and W_1 the load at the joint. Then the stress in the vertical is

$$S=W_1-\frac{S_h}{p}(h-h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

If $W_1>S_h(h-h_1)/p$, S is plus, and the stress is tensile; otherwise it is compressive.

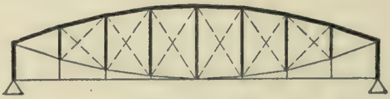


FIG. 27.

Analyzing the forces meeting at an upper chord joint, and letting h' and h_1' denote the vertical projections of the chord members to either side of the joint, and W' the load at that joint,

$$S=\frac{S_h}{p}(h'-h_1')-W'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If $W'<S_h(h'-h_1')/p$, S is plus and the stress is tensile; otherwise it is compressive. The values of S from Eqs. (58) and (59) must, of course, be equal, and as these equations are written, the sign of S in any given case will be the same from either.

For the usual case of parabolic arcs of equal curvature, Fig. 27, $h=h'$ and $h_1=h_1'$, and combining Eqs. (58) and (59)

$$S=\tfrac{1}{2}(W_1-W'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

which is then the stress in every vertical. For the dead load in a through bridge, $W_1>W'$ and the stress in the verticals is tensile, whereas in a deck bridge the reverse is true. If $W_1 = W'$ the stress in the verticals is zero. For a uniform live load covering the entire bridge, the stress in the verticals is tensile for a through bridge (in Eq. (59) $W' = 0$), and compressive in a deck bridge (in Eq. (58) $W_1 = 0$). In the former case, these tensile stresses are the maximum stresses in the verticals. In either case the compressive stresses become maximum when the load extends over the longer segment of the truss from each vertical. The exact position for maximum compressive stress in any vertical and tensile stress in any diagonal, as well as the values of these stresses, may be found by the methods explained in previous articles. The use of this truss is restricted, in general, to highway bridges, for which the stresses may most suitably be computed by the conventional method for uniform loading. Since the stresses in the diagonals are zero when the truss is fully loaded, it follows that if only a single diagonal exists in each panel, the maximum tensile stress, due to the loading on a certain part of the truss, must be equal to the maximum compressive stress, due to the loading on the remaining part of the truss.

If both chords are of equal curvature, the lever arms of the two diagonals in any panel about their common moment center are equal. Therefore, if any load tends to produce a compressive stress in one diagonal which that member is unable to resist, the other diagonal will be subjected to a tensile stress of like magnitude. Again, in the case of parabolic arcs of equal curvature, any variations in the greatest height of the truss between chord centers will cause proportionate changes in all other chord intercepts. Therefore, as for the parabolic truss in the preceding article, the stresses in the verticals will not be affected, whereas the stress in any diagonal will be altered in proportion to the change in the secant of the angle which it makes with the vertical.

Example. Required the maximum live-load stresses by the conventional method in the truss shown in Fig. 28, from a uniform load of 1000 lbs. per foot, per truss.

Solution. All loads and stresses will be expressed in thousands of pounds.

$$\text{Panel load} = 25 \times 1.0 = 25.0.$$

From Art. 144,

$$S_h = \frac{wl^2}{8h_{\max}} = \frac{1 \times 200^2}{8 \times 32} = 156.25.$$

CHORD STRESSES

$$AB(=Ab)=\frac{156.25}{25}\times 25.96=162.3;$$

$$BC\,(=bc)=\quad \text{“}\quad \times 25.50=159.4;$$

$$CD\,(=cd)=\quad \text{“}\quad \times 25.18=157.4;$$

$$DE\,(=de)=\quad \text{“}\quad \times 25.02=156.4.$$

The upper-chord stresses are compressive; the lower, tensile.

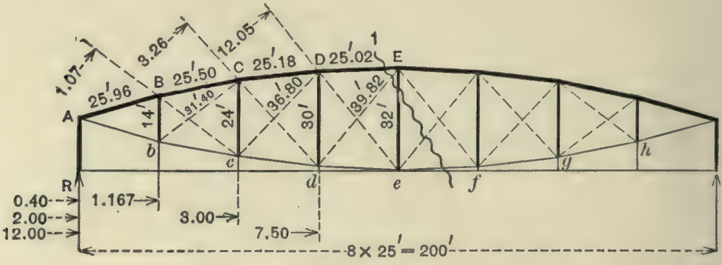


FIG. 28.

WEB STRESSES

(*a*) *Lever Arms.* All lever arms will be expressed in panel lengths The horizontal distances from the left reaction R to the intersection of the chord members in successive panels are then:

$$\frac{14}{24-14}-1=0.40 \qquad \frac{24}{30-24}-2=2.00 \qquad \frac{30}{32-30}-3=12.00.$$

These values are indicated in the figure to the left of R.

The lever arms of the diagonals by Eq. (12), Art. 131, are:

$$Bc \text{ and } Cb,\ (0.4+1)\times\frac{24}{31.40}=1.07;$$

$$Cd \text{ and } Dc,\quad (2+2)\times\frac{30}{36.80}=3.26;$$

$$De \text{ and } Ed,\ (12+3)\times\frac{32}{39.82}=12.05.$$

Since one of the diagonals in each panel is in action under partial loading, the moment centers of the verticals lie at the intersections of the upper and lower chord members in adjacent panels. These lever arms are:

$$Bb,\ \frac{14}{7+5}=1.167 \quad Cc,\ \frac{24}{5+3}=3.00 \quad Dd,\ \frac{30}{3+1}=7.50.$$

(*b*) *Diagonals.* For maximum tension in *Bc*, for example, full panel loads, 25.0, must be applied at every joint from *c*, inclusive, to the right. Thus,

$$R \times v/z = \textit{stress}$$

$$Bc=\frac{25}{8}\times 21=65.6\times\frac{0.40}{1.07}=(+)24.5;$$

$$Cd=\text{“}\times 15=46.9\times\frac{2.00}{3.26}=(+)28.8;$$

$$De=\text{“}\times 10=31.25\times\frac{12.00}{12.05}=(+)31.1.$$

If the diagonal *Cb* did not exist, a load at *b* would produce a compressive stress in diagonal *Bc*. That stress must be equal to the tensile stress, 24.5 (for loads from *c* to the right), since stress *Bc* is zero for the bridge fully loaded. But if *Bc* is flexible, so that it can act only in tension, and the other diagonal *Cb* is provided, any load tending to produce a compressive stress of 24.5 in *Bc* will develop instead a tensile stress of like magnitude in *Cb*. Hence

$$Cb=Bc=(+)24.5;$$

$$Dc=Cd=(+)28.8;$$

$$Ed=De=(+)31.1.$$

(*c*) *Verticals.* For maximum compression in *Bb*, for example, the same joints must be loaded as for maximum tension in *Bc*. Hence by using the values of *R* derived for the stresses in the diagonals, and the proper lever arms for *R* and the verticals:

$$Bb=65.6\times\frac{1.167-1}{1.167}=(-)9.4;$$

$$Cc=46.9\times\frac{3-2}{3}=(-)15.6;$$

$$Dd=31.25\times\frac{7.5-3}{7.5}=(-)18.8.$$

Stress *Ee* is equal to the left reaction *R* for full panel loads at *f*, *g* and *h*, as may be seen by taking section 1 and considering the equilibrium of the vertical forces to the left. Chord stresses *DE* and *ef* are equal, since their moment centers lie in the same vertical plane at *e* and *E* respectively, so that their vertical components will be equal and opposite. Thus

$$Ee=\frac{25}{8}\times 6=(-)18.75.$$

The tensile stress in the verticals is by Eq. (60), $25/2 = 12.5$, and the tensile stress in the vertical hangers below the bottom chord is 25.0. The compressive stress in the vertical end post is $25.0 \times 3.5 = 87.5$.

From the above it is seen that after the lever arms and S_h have been determined, the computation of all the maximum live stresses involves in this case only eleven simple numeric operations, namely four for the chords, three for the diagonals and four for the verticals.

146. Stresses in Web Members whose Moment Centers Lie between the End Supports. Considerations of economy in the design of simple trusses require that for moderate span lengths they shall be of uniform height, and that for greater span lengths the height shall increase from the ends toward the center. Trusses in which the height between chord centers varies in the opposite manner, as in Fig. 29, are uneconomical, but they are sometimes used for æsthetic reasons.

Consider the stress in any diagonal, as Cd, for example, whose moment center, G, lies between the end supports. Assume a load applied between A and C, to the left of section 1. Considering the moments about G of the forces to the right of the section, the moment of R_1 is counter-clockwise, so that the stress in Cd must be exerted, with reference to d, in the direction of the lower arrow, which indicates a tensile stress in that member.

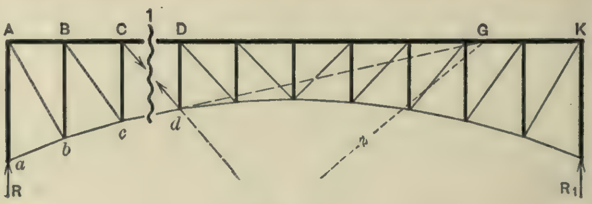


FIG. 29.

Applying a load at any point to the right, between D and K, and taking moments about G of the forces to the left of the section, the moment of R is clockwise, so that the stress in Cd must act, with reference to C, in the direction of the upper arrow. This again indicates a tensile stress in the diagonal. Further, since loads either at C or D cause tension in Cd, a load anywhere in panel CD induces a tensile stress in that member. Thus, it is seen, that *the stress in a web member whose moment center lies* *between the end supports becomes maximum when the load extends over the entire span.* Since the tensile stress in such a diagonal is not subject to reversal, no counter is required in the corresponding panel.

It may be shown similarly that if the moment center G should lie to the right of K, a load placed anywhere between D and K will produce tension in Cd, as before, whereas a load at any point from A to C will cause compression in that member. A load in panel CD will produce no stress in Cd if it is so placed that the moment about G of the floor-beam reaction at C equals that of the truss reaction, R, at a. A load in panel CD to the left of the point thus located will produce compression in Cd, and a load to the right, tension. The exact position of a concentrated-load system for maximum stress in any web member, and the corresponding stress, may be found in this case by the methods explained in previous articles for trusses whose height increases from the ends toward the center.

If the moment center G should happen to lie vertically above the right support, loads to the left of the section will produce no stress in Cd, since the moment of R_1, the only force to the right of the section, is then zero. Loads to the right of the section will cause tension in the diagonal, as before.

Again, since for web members to the left of the center the lever arm of R with respect to G is in general much longer than that of R_1, it follows that the stresses in such members from loads to their right are proportionally greater than from corresponding loads to their left, at the same distance from the center of the truss.

Problem. Show that the relations mentioned in the last paragraph are reversed for web members in trusses whose height increases from the ends toward the center.

147. Trusses with Multiple Web Systems.

The ambiguity of stress in trusses with multiple web systems is greatly increased if one or both chords are inclined. The assumption in the analysis of such trusses with horizontal chords that the web members of each component truss are affected only by the loads at the joints of that truss, is wholly inadmissible for trusses with inclined chords. Thus, in Fig. 30, a load at d, for example, will produce stresses in all chord members. If the inclination of members BC and CD is not equal, the forces meeting at C cannot be in equilibrium if the stresses in the web members are zero; for, to satisfy $\Sigma H=0$, the horizontal components of the two chord stresses must be equal, but if their slope is different, their vertical components will be unequal, so that $\Sigma V=0$ cannot be satisfied. The stresses in trusses of this type cannot be satisfactorily analyzed by the principles of statics. The use of such trusses is exceptional, and it is desirable that it should be wholly abandoned.

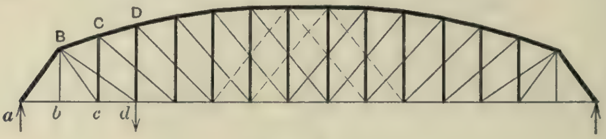


FIG. 30.

CHAPTER IX

MODERN BRIDGE TRUSSES AND THEIR DEVELOPMENT

A. HISTORICAL NOTES

148. Early History. The modern bridge truss, regarded as a frame composed of an assemblage of triangles, without unnecessary members, is essentially a product of the last century, during which its development took place concurrently with that of the theory of stresses. Its evolution from its earliest prototypes cannot be definitely traced. It is reasonable to suppose, however, as is commonly assumed, that the first trusses were simple triangular roof frames in which the rafters were connected by horizontal ties to relieve the thrust on the walls. The earliest bridge truss was probably the *king-post truss*, Fig. 1 (*a*), formed by adding a vertical tie (or post, in the case of an inverted deck truss) to a triangular frame to stiffen and strengthen a continuous horizontal beam. This was probably followed by the *queen-post truss*, Fig. (*b*), for longer spans. In the latter, unlike the former, the lower chord must necessarily be continuous—i.e., capable of resisting bending moments throughout its length—in order that the frame, which is not strictly a truss, may be stable. The next step in the development of roof trusses led probably to the type shown in Fig. (*c*), designed to give additional support to long rafters. Trusses of this form are known to have been used by the Egyptians and Romans.

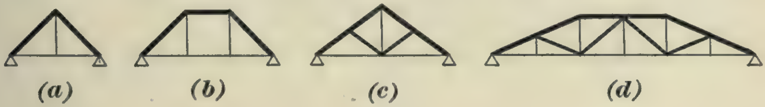


FIG. 1.

Palladio's remarkable treatise on architecture, published in 1570, contains a drawing of the Cismone Bridge, which had a span of 114 ft., in which the trusses were of the type shown in Fig. (*d*). Correct methods for determining the stresses in a truss did not become known, however, till several centuries later. Bridge trusses with triangular subdivisions conforming to correct principles of design were rarely used before the nineteenth century.

The most notable achievements in bridge building in ancient times were those of the Romans in the construction of stone arches for aqueducts and bridges. Some marvelous examples of these structures have survived in various degrees of preservation to this day. The remains of stone piers built by the Romans to support wooden bridges of uncertain design have also been found at various localities. A bas-relief on the Trajan column in Rome represents an arch bridge built across the Danube about 104 A.D., of unknown span and apparently of wood. In ancient and medieval times ship bridges—the forerunners of modern pontoon bridges—in which the roadway was supported by ships held in place by ropes, chains, and anchors, were largely used for crossing wide rivers, especially in military operations.

149. Early American Wooden Bridges. In the development of long-span wooden bridge construction, toward the close of the eighteenth and early in the nineteenth century, American bridge builders took a leading part. Among these Timothy Palmer, Theodore Burr and Lewis Wernwag stood out most prominently. Since the analysis of stresses was yet unknown, sound judgment, originality and daring had to make up for the lack of theory.

In 1792 Palmer built the Essex-Merrimac Bridge over the Merrimac River near Newburyport, Mass. The longest span was 160 ft., which was bridged by arched wooden triangular trusses reinforced by wooden braces from the piers to the lower chord. In 1794 he built a bridge across the Piscataqua River, in which the chord of the longest arched truss was 244 ft. 6 ins. Among the many bridges built by Palmer is the one over the Delaware River at Easton, Pa., shown in Fig. 2,* which was completed in 1806. This bridge, which included three 163-ft. clear spans, was completely enclosed with boards to protect it against the weather, and was in service till 1895, i.e., for eighty-nine years.

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* This figure is based on a drawing prepared under the direction of Professor J. M. Porter, from measurements of the bridge.

In 1804 Burr built the Waterford Bridge, shown in Fig. 3, over the Hudson River. This bridge embraced four spans varying from 154 to 180 ft. in the clear. The design was of a composite nature, consisting of the combination of a truss with an arch. This bridge was converted into a ' covered bridge ' by the addition of weather boarding in 1814, and remained in service until its destruction by fire in 1909.

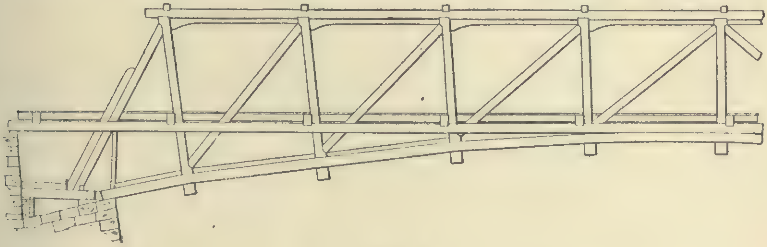

FIG. 2.

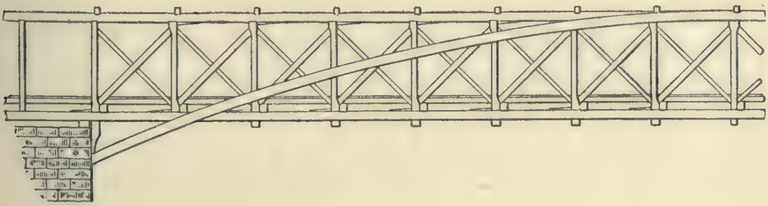

FIG. 3.

The Colossus Bridge over the Schuylkill River at Fairmount, Philadelphia, shown in Fig. 4, was built by Wernwag in 1812.

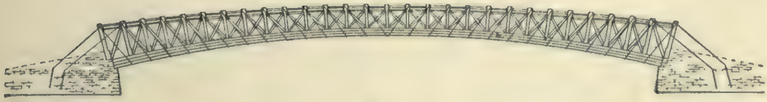

FIG. 4.

This bridge had a clear span of 340 ft. $3\frac{3}{4}$ ins., and was remarkable for the boldness of its design as well as its unprecedented span length. As seen in the figure, this structure was a trussed arch in which the horizontal thrust at the supports was resisted by heavy abutments. To Wernwag also fell the distinction of building the first wooden railroad bridge, in 1830 at Monoguay, on the Baltimore and Ohio Railroad.

The wooden lattice bridge, Fig. 5, patented by Ithiel Town in 1820, marked an important step in the evolution of modern bridge trusses. This type and its derivatives came into extensive use in this and foreign countries, and are employed to this day for wooden trusses,* and in modified form with fewer web systems, for riveted steel trusses of the multi-triangular type, known also as 'lattice trusses' or 'lattice girders.'

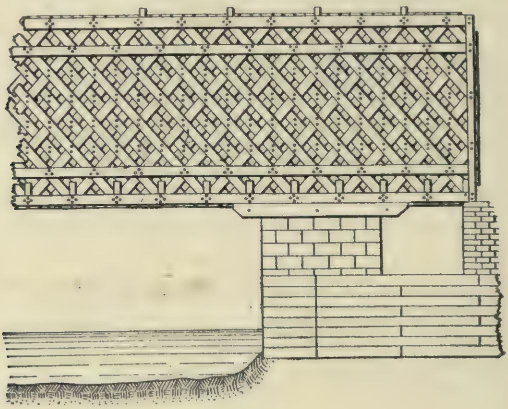

FIG. 5.

In 1830 Long patented a truss in which the web members were disposed as in the Burr truss, but in which the auxiliary arch was omitted. Many trusses both of the Long and Town type were built, however, in combination with arches, especially for longer spans.

In 1840 the Howe truss, Fig. 6, was patented by William Howe. The members of this truss were arranged as in the Long truss, but while the latter was made wholly of wood, the verticals in the Howe truss consisted of iron rods with screw ends which admitted of adjustment by means of nuts. By tightening these nuts initial

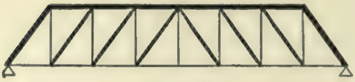

FIG. 6.

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* See footnote, Art. 151.

stresses could be produced in the entire web system. The objectionable features of the Long truss — namely, looseness of the web members resulting from unequal shrinkage of the timbers, and the difficulty of providing efficient connections for wooden tension members—were thus successfully overcome. The Howe truss has been used more extensively than any other type for wooden bridges, on account of its general excellence and low cost. Long usage has led to many improvements in details, and where suitable timber is abundant and low first cost is an important consideration, the wooden Howe truss is used to this day.

In 1844 the Pratt truss, Fig. 7, was patented by Thos. W. and Caleb Pratt. It differs from the Howe truss in that the verticals are designed to act in compression and the diagonals in tension, instead of the reverse. As originally constructed with the diagonals of wrought iron and the other members of wood, it was more expensive than the Howe truss and could not compete successfully with the latter; but as executed later in iron and steel, it came into wider use than any other type, and holds first rank among bridge trusses at the present time.

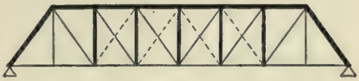

FIG. 7.

150. Early Iron Bridges. *Arch Bridges.* The first iron bridge on record was designed and built by Abraham Darby, in 1776–79, across the Severn River, at Coalbrookdale, England. This bridge, which is in service at this day—132 years after its erection—has a span of 102 ft., and consists of five cast-iron arches of the form shown in Fig. 8,* with iron plate flooring. Following the successful construction of this bridge many cast-iron arch bridges were built, notably in England.

Suspension Bridges. Prior to 1776, bridge building had been carried on wholly in stone and timber with the exception of certain crude beginnings in the construction of suspension bridges in the form of so-called 'chain bridges,' in which iron chains extending across towers to suitable anchorages furnished support to wooden floor systems. In 1796 James Finley built the first bridge of this type in America, across Jacob's Creek, between

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* From Mehrtens' Eisenbrückenbau, Vol. I.

Uniontown and Greensburg, Pa. Among the many chain bridges erected by Finley and others after his method, the largest was that across the Schuylkill River, at Falls of Schuylkill, near Philadelphia, which had two spans of 153 ft. This bridge, built in 1808–09, after Finley's general system, though not by himself, was imperfectly designed and failed under the weight of a drove of cattle in 1811. It was reconstructed, but in 1816 it failed again under a load of snow and ice. It was then replaced by a temporary foot-bridge, of about 407 ft., supported by wire cables. This bridge, erected in 1816 by White and Hazard, wire manufacturers in that locality, is the first wire-cable suspension bridge on record. Beginning five years later this form of bridge construction was greatly developed in France and Switzerland.

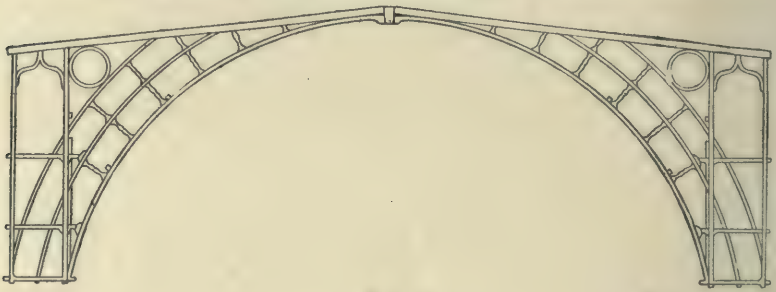

FIG. 8.

The Essex-Merrimac chain suspension bridge over the Merrimac River near Newburyport, Mass., built by John Templeman after the Finley method, has a remarkable history. This bridge had a span of 244 ft. and was erected in 1810 to replace in part the wooden bridge built by Palmer in 1792. It failed in 1827 under the combined weight of snow and traffic, but after repairs and strengthening it remained in continuous service till 1909, when it was replaced by a wire-cable suspension bridge.

In 1841 Charles Ellet completed a wire-cable suspension bridge, with a 343-ft. span, across the Schuylkill River at Philadelphia, which was the second bridge of this kind in America. In 1847 he built a bridge of this type across the Ohio River at Wheeling. This bridge had the unprecedented span length of 1010 ft. The means used for stiffening the roadway were inadequate, however, and the bridge was destroyed by a wind storm in 1854.

In 1855 John A. Roebling built his famous wire-cable suspension bridge across the Niagara gorge. This bridge had a central span of 821 ft. 4 in., and carried a double-floor system, the upper floor supporting a single-track railway and the lower a highway. This was the first permanent railway bridge of the suspension type. A stiffening truss of the Pratt type, 18 ft. deep, was an important feature of this structure, which embodied other remarkable improvements.

Truss Bridges. The construction of iron truss bridges was not begun till about 1840, and did not assume important proportions till after the publication in 1847 of Squire Whipple's notable work entitled "An Essay on Bridge Building," in which the analysis of the stresses in a truss was for the first time correctly developed. Prior to that time the important functions of the web members were but imperfectly understood, and it was commonly supposed that the strength of a truss was dependent essentially on the chords. In the early Howe trusses, for example, the vertical ties were made of the same size throughout the truss.

The first iron truss bridge in America was built by Earl Trumbull over Erie Canal in 1840. It consisted of cast-iron trusses combined with a wrought-iron suspension system.

The unsuitability of cast-iron for tension members, on account of its brittleness and low tensile strength, was, however, soon apparent. In fact in the same year, 1840, Whipple built a 72-ft. bowstring truss bridge in which the tension members were all made of wrought iron and the compression members of cast iron.

The first iron truss railroad bridge in America was designed by Richard B. Osborne and built in 1845, over a small stream near Manayunk, Pa., on the Philadelphia & Reading Railroad. It was a double-track bridge with three trusses of the Howe type, Fig. 9,* and it had a clear span of 34.2 ft. Each chord was composed of a pair of rolled square wrought-iron bars. The verticals were also made of that material and the diagonals of cast-iron. The floor consisted of wooden floor beams resting on the lower chords and supporting wooden stringers directly under the rails. This bridge remained in service until 1901, although for many years before its removal it had been supported at intermediate points by wooden trestles. The trusses of this historic bridge are preserved in the company's store yards at Reading, Pa.

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* This figure is based on a drawing prepared under the direction of Mr. Edwin F. Smith from measurements of the bridge.

The construction of iron truss bridges at this period was greatly stimulated by the development of the railroads, but a number of disastrous failures served to bring these structures into such disrepute that in 1850 all iron bridges on the Erie Railroad were replaced by wooden ones.

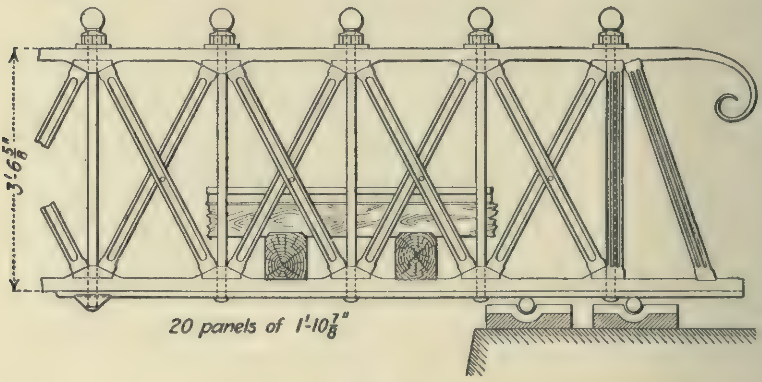

FIG. 9.

The decision at about this period of Benjamin H. Latrobe, Chief Engineer of the Baltimore & Ohio Railroad, to use only iron bridges on the extension of that road from Cumberland, Md., to Wheeling, W. Va., was an important step toward the general adoption of iron for railroad bridges. The bridges built under Latrobe's direction were largely of the Bollman and Fink types.

Bollman Truss. In the Bollman truss, Fig. 10, the load at each panel point is carried to the vertical end posts by a pair of diagonals. The lower chord receives no stress from vertical loading. The members represented by broken lines are redundant, and were used to secure greater rigidity. Many bridges of this type, with cast-iron compression members and wrought-iron tension members, were built from about 1845 to 1855.

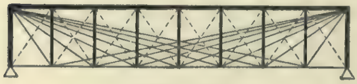

FIG. 10.

Fink Truss. The characteristic features of the Fink truss may be seen more clearly in the deck form, Fig. 11 (*a*), than in the through form, Fig. (*b*). In the latter the stress in the lower chord from vertical loading is zero, and the web members distinguished by broken lines are redundant, as in the Bollman truss.

Considering a load, W, at the first panel point of an eight-panel truss, for example, $\frac{1}{2}W$ is transmitted to the nearer support and $\frac{1}{2}W$ to the quarter-point of the truss. Of the latter, $\frac{1}{4}W$ is conveyed to the nearer support and $\frac{1}{4}W$ to the middle of the truss, whence $\frac{1}{8}W$ is carried to each support. The reaction at the nearer support is therefore $\frac{1}{2}W+\frac{1}{4}W+\frac{1}{8}W=\frac{7}{8}W$, and the reaction at the farther support is $\frac{1}{8}W$.

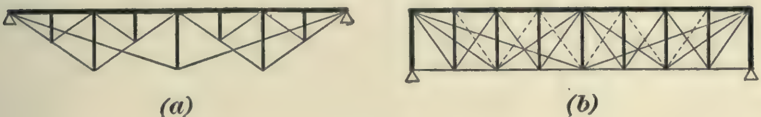

FIG. 11.

The Fink truss was an improvement on the Bollman truss in that it lessened the objectionable concentration of numerous web members at the end joints, and reduced the number of excessively long tension web members with unfavorable angles of inclination.

Fink trusses were first adopted for a through bridge of three 205-ft. spans, built in 1851–2, across the Monongahela River at Fairmont, W. Va., on the Baltimore & Ohio Railroad. Many bridges of this type, especially deck bridges, were built during the ensuing thirty years, and it was used for viaducts as late as 1886. A railroad bridge, built in 1871, across the Missouri River at St. Charles, Mo., contained the longest Fink trusses ever constructed, the span having a length of 306 ft. 6 in.

In comparison with modern bridge trusses the Bollman and Fink trusses are uneconomical and deficient in rigidity. The Fink truss is well adapted, however, to roof trusses (Art. 63), for which it is largely used at this day.

Whipple Truss. In the Whipple truss, Fig. 12, invented by Squire Whipple, the diagonals extend over two panels. This arrangement was designed to secure favorable inclinations of the diagonals for trusses with heights suitable for longer spans, without sacrificing the economic advantage of short panels for the floor system.

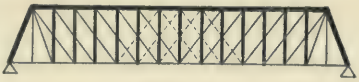

FIG. 12.

The first iron bridge of the Whipple type was built by Whipple in 1852–3, on the Rensselaer & Saratoga Railroad, near Troy, N. Y. This bridge had a clear span of 146 ft., and remained in service till 1883.

The Whipple truss grew rapidly in favor and became in time the leading truss for spans above 175 ft. In the earlier days Whipple trusses were used also for much shorter spans and were not infrequently built with diagonals extending over three and even four panels, and with panel lengths of 5½ to 7 ft.* Whipple trusses were used in 1877 for the 515-ft. span of the Ohio River Bridge on the Cincinnati Southern Railway, and reached their greatest length in the two 518-ft. spans of the bridge built in 1882 over the same river at Cairo, Ill. Comparatively few bridges of this type have been erected since about 1886. Whipple trusses are now rarely built, since they are less economical than trusses of more modern types, in which the stresses are, moreover, statically determinate.

The Post Truss was invented by Simeon S. Post, who built the first bridge of this type on the Erie Railroad in 1865. As seen in Fig. 13, the end panels of the lower chord are half as long as the other panels, and the arrangement of the web system is such that horizontally the posts extend over a half-panel, and the main diagonals over one and one-half panels. The counters, represented by broken lines, were inserted in every panel and brought under initial stress to increase the rigidity of the web systems. Many bridges of this type were built for ten or more years after its introduction, although the stresses are statically indeterminate and the supposed economic advantage is really fictitious.

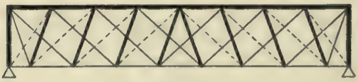

Fig. 13.

Early Long-span Bridges. The Steubenville Bridge, built by Linville, in 1862, across the Ohio River, contained a channel span of 319 ft., with trusses of the Whipple type. Until that year the 205-ft. Fink trusses of the Monongahela River Bridge, previously mentioned, were the longest iron bridge trusses in America.

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* Report of Board of Railroad Commissioners of the State of New York on *Strains on Railroad Bridges* (1891).

The Louisville Bridge across the Ohio River, built by Fink in 1870, contained a channel span with a length of about 396 ft., of which the trusses were of the triangular type, with sub-panels, shown in Fig. 22 (*a*), Art. 154, but with horizontal chords.

The Ohio River Bridge, built in 1877 by Linville, contained a 515-ft. span. This bridge, previously referred to, was the first truss bridge with a span over 500 ft., in length. Since the erection of this bridge many simple truss bridges have been built with spans ranging from 500 to 546.5 ft., and the upper limit has been exceeded in two instances. The St. Louis Municipal Bridge, now in course of construction, contains three spans of 668 ft. The longest truss highway bridge has a span of 586 ft. This bridge was built in 1904 across the Miami River at Elizabethtown, Ohio.

B. MODERN TRUSS BRIDGES

151. General Classification of Truss Bridges. Bridges are classed as steel, combination, or wooden bridges, according to the materials of which they are made. A bridge made wholly of wood or one in which steel is used only for the tension web members is called a wooden bridge. A ' combination bridge ' is one in which the lower chords as well as the tension web members are made of steel, and the compression members of wood. Combination bridges were used to a considerable extent when iron was more costly and wood cheaper than at present, but they are rarely employed at this day.

Steel bridges are distinguished as *pin connected* or *riveted bridges*, according to whether the members are connected at each joint by a single pin or by rivets. The use of forged eye-bars, Fig. 14 (*a*), for the principal tension members and pins, Fig. (*b*), for connections are characteristic features of American practice in the design of larger bridges. The pin is so placed that its axis passes through the common point of intersection of the axes of all members meeting at a joint, which ensures the greatest freedom from so-called ' secondary stresses' due to eccentric con-

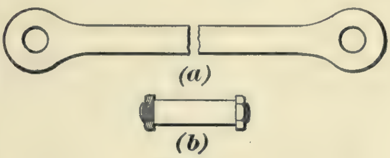

FIG. 14.

nections or to bending moments induced by resistances at the joint to the free angular movement of the members as the truss deflects. With pin connections eccentric stresses are almost wholly avoided, and the effect of bending is limited to that of the frictional resistances at the pin bearings. Pin connections afford further advantages over riveted ones in the facility and speed with which erection operations may be conducted—a matter of great importance in bridging streams subject to sudden floods which might endanger false-works.

In his treatise on bridge building, published in 1847, Whipple proposed modified details for his earlier bowstring canal bridges to adapt them to railroad requirements, in which he recommended pin connections for the upper-chord joints. The lower chords of his canal bridges consisted of wrought-iron links, which passed around wrought-iron trunnions at the joints.

The first bridge in which pins were used throughout was of the double-intersection Whipple type, built by Murphy in 1858–9. A bridge of the same type was built in 1861 by J. H. Linville, in which tension members of forged eye-bars were used for the first time in a truss bridge, although they had previously been employed to some extent in European practice, especially in the chains of suspension bridges.

152. Materials. The developments of the nineteenth century in bridge building are largely attributable first, to the invention of the steam engine and its application to railroads and manufacturing operations; and second, to advance in metallurgical processes. Until about the beginning of that century wood and stone were the only materials commercially available for bridges. Then cast iron began to be used for arch bridges and solid I-beam girders, and later, in combination with wrought iron, for truss bridges.

The first bridges in America in which wrought iron was used throughout were introduced in 1859 by Howard Carroll on the New York Central Railroad. These bridges were of the lattice type. In 1863 J. W. Murphy built a bridge of the Whipple type across the Lehigh River at Mauch Chunk on the Lehigh Valley Railroad, in which wrought-iron was used for the first time in a bridge of this type for compression members as well as tension members. The compression members were connected, however, by cast-iron joint boxes, a practice which was widely followed till its abandonment in 1876 for railroad bridges and about 1890 for highway bridges.

Steel was first used for structural purposes in certain parts of the St. Louis Bridge across the Mississippi River, built by J. B. Eads and completed in 1874. The Glasgow Bridge over the Missouri River, completed in 1879, is the first bridge constructed wholly of steel. The use of wrought iron in bridges was entirely abandoned about 1894. For a few years before that time many bridges were constructed partly of steel and partly of wrought iron, and in some cases wholly of steel. The general adoption of steel for bridges was due to the invention of the open-hearth process of steel making, which was found to yield a much more uniform and reliable product than the Bessemer process by which the steel used in earlier bridges was made.

Within recent years the adaptabilities of various 'alloy steels' to structural purposes have received much attention. The first application of nickel steel to bridge construction was in connection with the Queensboro Bridge, across the East River at New York. In this bridge, completed in 1909, nickel steel was used for the principal tension members. This material was also used in certain parts of the Manhattan Bridge at New York, and it has been adopted more recently for all the principal truss members in the three 668-ft. spans of the Municipal Bridge now (1911) in course of construction across the Mississippi River at St. Louis, at an estimated saving of about one-third in the weight of the trusses. Unfortunately, at present prices (1911) the saving in weight by the use of nickel steel is largely, if not wholly, offset by its much greater cost per pound. Promising investigations have been conducted more recently with other alloy steels, notably nickel-chrome-vanadium steel, with a view of determining their suitability for bridge construction, and further important developments along these lines are to be anticipated.

Reinforced concrete has also come into extensive use within recent years for railroad as well as highway bridges. Beams and slabs are used for short spans, and arches for longer spans.

Timber is still used to some extent both for railroad and highway bridges. In certain New England districts where the cost of timber of suitable quality and size for larger bridges of the Howe type is prohibitive, Town lattice bridges have been used instead. Thus on the Boston & Maine Railroad, on which nearly half the new bridges built to replace old ones were still made of wood until about 1905, Town lattice bridges were used for spans from 60 to 150 ft. in length, at a cost about one-third less than that of steel structures, and even now bridges of this type are occasionally built on the northern lines. These bridges are built of spruce timber, covered in, and their life is estimated at forty to fifty years.* For spans above 110 ft. it is found economical, as well as conducive to rigidity, to build the trusses in combination with wooden arches, in much the same way as practiced nearly a century ago. The loads are conveyed to the arch through vertical rods spaced at intervals of a few feet, and the arch is designed to carry the entire live load in addition to its own weight, the truss serving chiefly to stiffen the arch, especially against unsymmetrical loading.

153. Modern Types of Steel Trusses for Railroad Bridges. The types of trusses commonly used for railroad bridges in what may be called standard American practice for heavy loading, will now be considered. The limiting span lengths to be given in that connection are to be regarded only as approximate values fairly indicative of approved current practice.

For spans below 110 ft. riveted plate girders, consisting of solid rolled web plates and flanges made up of angles and plates, are usually preferred to trusses, on account of their rigidity, ease of maintenance and general excellence. Plate girders are not infrequently used for spans up to 120 ft. or more.

Riveted trusses are usually preferred for spans below 160 ft., and pin-connected trusses for spans above 175 ft. Between these limits of 160 to 175 ft. there is no well-defined preference for either type, although there is a growing tendency to use riveted trusses for spans up to 175 or 200 ft., and they are sometimes used for much larger spans.

Triangular Truss. The triangular truss was devised by Neville, a Belgian engineer, in 1846. The details he proposed were improved by Warren, an Englishman, in 1849, and the 239-ft. span of the Trent Bridge was built in 1851, on what came to be known as the Warren system.

Riveted Triangular Trusses, or *Warren Trusses* with verticals,

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* *Wooden Bridge Construction on the Boston and Maine Railroad*, by J. P. Snow, *Journal Assoc. of Eng. Socs.*, Vol. XV (1895).

Fig. 15 (*a*), are commonly used for spans of 120 to 160 ft., since they are somewhat cheaper than riveted Pratt trusses and equally satisfactory. The main panels are sometimes divided into subpanels, as in Fig. (*b*), especially where solid ballasted floors render the use of relatively short panels desirable, or where the clearance requirements beneath the bridge necessitate the use of shallow floor girders.

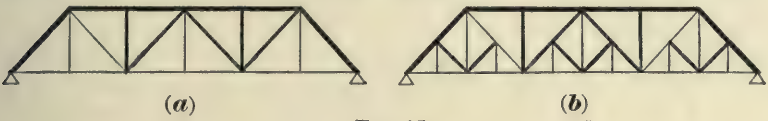

(*a*) (*b*)

FIG. 15.

Pratt Truss. The Pratt truss with horizontal chords, Fig. 16 (*a*), is also used for riveted trusses up to 160 ft. In railroad bridgs the counters in such riveted trusses are frequently omitted and instead the diagonals subject to counter stresses are counterbraced. This practice is to be recommended as conducive to rigidity and determinateness of stress. Subpanels, as shown in Fig. (*b*), are sometimes used for reasons stated above in connection with the triangular truss.

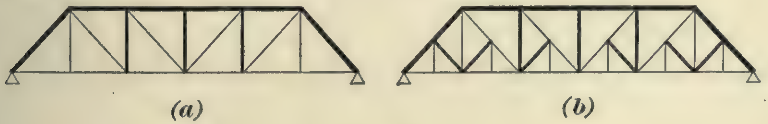

(*a*) (*b*)

FIG. 16.

The Pratt truss with pin connections is commonly used for through bridges for spans from 160 to 175 ft., and less frequently for spans from 150 ft. and under to 160 ft. The Pratt truss is better adapted to pin connections than the triangular truss.

In the case of deck bridges, the upper chord must be horizontal, and to meet clearance requirements underneath it is usually necessary that the lower chord should also be horizontal. Under these circumstances the pin-connected Pratt truss is commonly used for spans from 160 to 250 ft.

Multi-triangular Trusses. Riveted trusses with double, triple and quadruple web systems, Fig. 17, are sometimes, though now rarely, used for spans of moderate length, up to about 150 ft. Trusses of this type, known also as *lattice trusses*, or *lattice girders,* were evolved from the simple Warren truss at the one extremity and the Town truss, Fig. 5, Art. 149, at the other. Although these trusses are in general statically indeterminate, the stresses may be found with sufficient accuracy for all practical purposes by simple methods based on the principles of statics, as has been seen in Chapter VII. A double-triangular web system may be made statically determinate by the unsymmetrical arrangement shown in Fig. (*b*). The chord stresses from vertical loading in the half-panels at the right end are then zero, and the vertical components of the stresses in the diagonals meeting at that end are each equal to one-half the right reaction. The stresses in the other members may be found by proceeding from the right end and analyzing the stresses at successive joints.

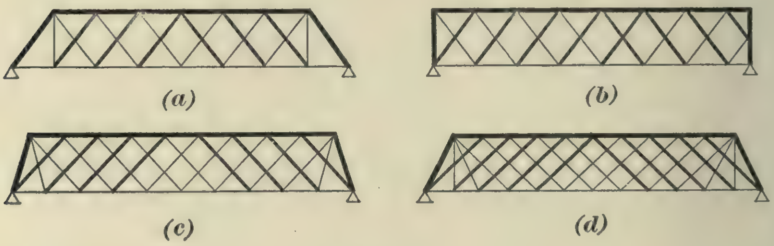

(*a*) (*b*)
(*c*) (*d*)

FIG. 17.

Riveted Whipple trusses (Art. 150, Fig. 12), are also used occasionally, though very rarely.

Pratt Truss with Inclined Chords. The pin-connected Pratt truss with inclined chords, Fig. 18, is commonly used for through bridges for spans from 175 to 300 ft. For spans greater than about 175 ft. it is more advantageous to use inclined, or so-called 'curved' chords, than horizontal chords, because (1) the reduction of material in the web members through decreased stresses and lengths exceeds the increase in the chords through opposite tendencies; (2) the maximum stresses in the chords are then nearly uniform, which is a desirable feature, especially for the economic design of the compression chord; (3) the reduction in height toward the ends conduces to stability against the overturning effect of wind-pressure, and lessens

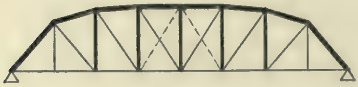


FIG. 18.

the wind-load stresses in the chords, as will be seen later; and (4) a greater central depth of truss may then be used to advantage, which effects a saving of material in the chord members near the middle of the truss.

Pratt Truss with Sub-Panels. The economic depth of a truss ranges from about one-fourth to one-sixth the span length, and the economic panel length from about one-fifth to one-sixteenth the span length, for spans of 110 to 600 ft., respectively. That is to say, as the span increases, the economic depth of truss increases much more rapidly than the economic length of panel. The inclination of the diagonals of single intersection trusses becomes steeper therefore, as the span increases, whereas for economy of material, and also for convenient and efficient details at joints, it is desirable that the angle of inclination of these members with the horizontal should not be much greater than 50° (slope 1.2 on 1). To meet this requirement for larger trusses without exceeding the economic panel length the main diagonals are extended over a double panel, i.e., the main panel is subdivided into a pair of 'sub-panels' as in Fig. 19. The intermediate floor beams at the middle of the main panels are supported by a system of sub-verticals and sub-diagonals as seen in the figure. The unloaded chord is also frequently supported at the central points of the principal panels by means of sub-hangers (not shown in the figure) to reduce the bending stresses in the eye-bars from their own weight, and for various practical considerations, such as the maximum practicable length of eye-bars in the chords, the width of the bridge as affecting the slope of the diagonals in the lower lateral system, etc.

The Pratt truss with horizontal chords and sub-panels, Fig. 19, known also as the *Baltimore truss*, is well adapted to deck bridges for spans above 250 ft. Where clearance requirements admit, it is more economical in so far as the steel work is concerned, to use an inclined lower chord. The saving is, however, more than offset by the added cost of the masonry (Art. 142), and new deck bridges are therefore usually built with horizontal chords. Formerly this truss was used not infrequently for through bridges of longer spans, but it is econonomically inferior to similar

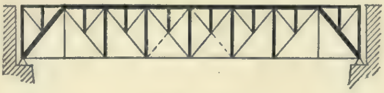


FIG. 19.

trusses with inclined upper chords and possesses no advantage over the latter.

The sub-diagonals used for the trussing of the double panels may be also arranged as in Fig. 16 (*b*). The triangular truss is also obviously adapted to either sub-system. Since in this truss the diagonals near the center are subject to reversal of stress, the use of pin-connections is undesirable (Art. 141).

Double-triangular Truss with Sub-Panels. This type of truss, shown in Fig. 20, has been employed to a considerable extent, but is now seldom used. It is inferior to the Pratt truss with sub-panels because the stresses are statically indeterminate and because it is not so well adapted to pin-connections. It is probably also somewhat less economical.

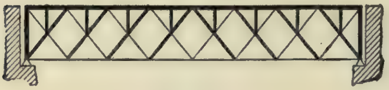


FIG. 20.

Pratt Truss with Inclined Chords and Sub-Panels. This type of truss, shown in Fig. 21, and known also as the *Pettit truss,* is used more extensively than any other for spans from 300 ft. to the highest limit to which the application of simple trusses has been carried, namely 668 ft. in the St. Louis Municipal Bridge. In Fig. (*a*) the sub-diagonals are ties and in Fig. (*b*) these members are struts. Both systems are largely used. The former is somewhat cheaper though perhaps less rigid than the latter. The connections are preferably of the pin type. If the inclined end post extends over a single panel, as in Fig. (*a*), the end diagonal should also be carried over only a single panel, on grounds of economy as well as good appearance. In the St. Louis Municipal Bridge the inclined end posts extend over two panels, as in Fig. (*b*), and four different panel lengths are used, ranging from 30 ft. at the ends to 48 ft. at the center, thereby securing favorable inclinations for the diagonals and improving the appearance of the truss. The intermediate horizontal sub-struts in Figs.

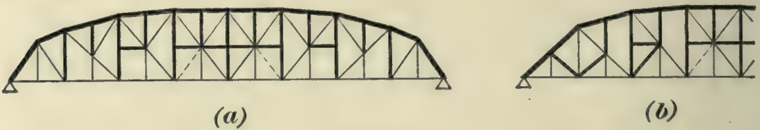


FIG. 21.

(*a*) and (*b*) receive no stress from vertical loading. Their purpose is to support the main posts against bending in the plane of the trusses. These posts are similarly stayed in vertical planes normal to the trusses, by transverse struts forming part of the ‘sway bracing.’

154. Modern Types of Steel Trusses for Highway Bridges. The design of a highway bridge is greatly affected by the live loading, which may range from about 80 lbs. per square foot of total floor surface, or a 6-ton traction engine, for ordinary country bridges, to a train of electric motor cars or electric freight cars weighing 2000 lbs. or more per foot of track, with concurrent loading of 100 lbs. per square foot on the remaining parts of the floor surface, for city bridges. The design is further influenced by the fact that in highway bridges the floors are not ‘open’ as in railroad bridges. In the latter each track is usually supported by only a pair of stringers, but in the former numerous stringers must be provided for the support of a solid pavement. The economic panel length for highway bridges is therefore much shorter than for railroad bridges. For the former the usual range is from 15 to 25 ft., and for the latter from 25 to 35 ft., the length increasing in a general way as the span and the depth of truss increase.

For highway bridges of the heaviest class the considerations governing the selection and general design of the main girders or trusses are essentially the same as for railroad bridges. For highway bridges of the average or lighter class the conditions are, however, quite different, and good practice may be defined in a general way as follows:

Rolled I-beams or plate girders may be used advantageously for spans up to 30 or 40 ft. For longer spans, up to 80 or 90 ft., pony trusses of the Pratt or triangular type may be used, for which riveted connections are decidedly preferable to pin connections, on account of their greater rigidity. For spans above 80 or 90 ft., through bridges with overhead lateral bracing should be adopted, and for spans up to 150 ft. riveted connections are preferable to pin connections. The trusses may be of the Pratt, triangular or double triangular type. For spans of from 165 to 225 ft. pin-connected Pratt trusses with inclined chords may be used to best advantage. For greater spans it is advisable to use sub-panels.

Where good appearance is a consideration bowstring trusses of the pony type, with parabolic upper chords, Fig. 26, Art. 144, may be used for short spans, and double parabolic, or 'lenticular' trusses, Fig. 27, Art. 145, for longer spans. The appearance of the latter is especially effective when there are three or more successive spans, as in Fig. 22. The objections to this truss are the comparative lightness of the web system and the restricted head room at the ends for portal bracing. Its use for short spans has been abandoned on account of its lack of rigidity when economically designed.

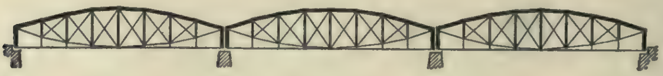
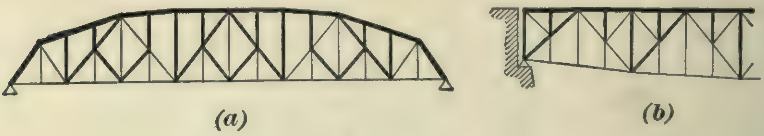


FIG. 22.

Triangular Truss with Inclined Chords and Sub-Panels. This truss, shown in Fig. 23, is sometimes used for long-span highway bridges. The truss is seen to present a good appearance, especially if the horizontal sub-struts are dispensed with, as in the figure. This may be done at little if any increased cost, since the long vertical posts are few in number and subject to comparatively small stresses, especially in the case of a through bridge as in Fig. (*a*). The saving in the sub-struts will nearly or quite offset the increase in the weight of the posts due to increased effective lengths.



(*a*) (*b*)

FIG. 23.

Comparative cost estimates between well-proportioned triangular and Pratt trusses with inclined chords and sub-panels, are apt to indicate a slight advantage in favor of the Pratt type, though that depends somewhat on the working stresses and other features of the specifications governing the design.

155. Characteristic Comparative Features of Current and Earlier Bridge Practice. The characteristic differences between current American bridge practice and that of 25 or 30 years ago are due principally to

1. The great increase in the weight of the live load for highway as well as railroad bridges (Chapter XII);
2. The substitution of steel for wrought iron;
3. Improved methods of manufacture;
4. Improved methods of erection.

In consequence of these influences, and from knowledge gained by observing the behavior of steel bridges under service conditions, modern bridge designs differ from earlier ones chiefly in

1. The increased span limits for which (*a*) plate girders are preferred to trusses and (*b*) riveted trusses to pin-connected trusses.
2. The avoidance of trusses with multiple web systems, or other elements tending to cause the stresses to become statically indeterminate.
3. The use of relatively deeper trusses, which effects a saving of material and increased stiffness.
4. The use of increased panel lengths and relatively deeper floor beams and stringers. This lessens the cost of workmanship and erection by reducing the number of joints, and is also conducive to rigidity.
5. The employment of riveted connections throughout the floor system, and 'stiff' lateral and transverse bracing with riveted connections, thus increasing the rigidity of the structure against vibrations.
6. The development and extensive use of greatly improved details.

Bibliographical Note. For valuable data of the history of American bridge building the reader is referred especially to *American Railroad Bridges*, by Theodore Cooper, *Trans. Am. Soc. C.E.*, Vol. XXI (1889), and the *The Evolution of the Practice of American Bridge Building*, by C. C. Schneider, *Trans. Am. Soc. C.E.*, Vol. LIV (1905).

The report of the Board of Railroad Commissioners of the State of New York, on "Strains on Railroad Bridges of the State" (1880 pp.), published in 1891, contains a wealth of interesting information concerning the twenty-five hundred railroad truss bridges of that State, of which many were of patterns then antiquated and now wholly obsolete.

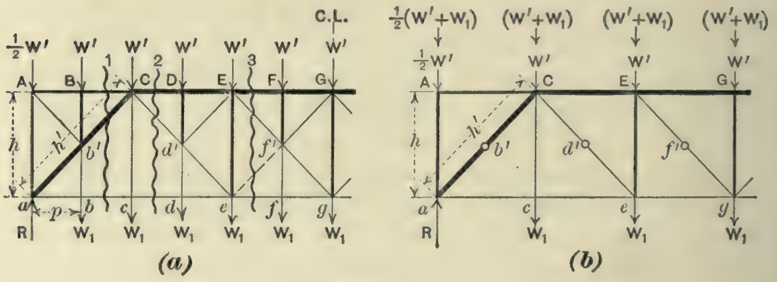
A comprehensive historical treatment of bridge building appears in Mehrtens' Eisenbrückenbau, Vol. I (1908).

CHAPTER X

STRESSES IN BRIDGE TRUSSES WITH SUB-PANELS

156. Pratt Truss with Sub-Panels (Baltimore Truss). Let a deck truss of 12 panels be considered, of which one-half is shown in Fig. 1 (*a*).

157. Dead-load Stresses. *First Method.* The intermediate joint loads W' at B, D and F, and W_1 at b, d and f are transmitted to b', d' and f' through the sub-verticals, and these loads may be regarded as conveyed thence to the upper chord joints by means of the triangular frames $Ab'C$, $Cd'E$ and $Ef'G$.



(*a*) (*b*)

FIG. 1.

The resulting redistribution of loading is as indicated in Fig. (*b*), and when this has been effected the ‘secondary’ members may be considered eliminated from the truss, leaving the ‘primary’ system shown in the figure, in the form of a simple Pratt truss.

The load $W'+W_1$ at d', for example, may also be assumed as transferred to joints E and e, through the frame $Ed'e$, and the loads at b' and f' as transferred in like manner to the upper and lower chord joints. The analysis of stresses on this assumption is more complicated, however, and leads to the same final results, as will be shown later (Art. 160).

(*a*) *Web Stresses.* The stresses in the secondary web members are evidently as follows:

Sub-struts, Bb', Dd', $Ff' = -W'$;
Hangers, $b'b$, $d'd$, $f'f = +W_1$;
Sub-diagonals, Ab', Ed', $Gf' = +\frac{1}{2}(W'+W_1)h'/h$.

The stress $+\frac{1}{2}(W'+W_1)h'/h$ is developed also in the upper halves Cb', Cd' and Ef' of the inclined end post and main diagonals in transferring the intermediate panel loads to the upper chord. This stress must therefore be combined with the stress in each of these members from the loading shown in Fig. (*b*). Similarly the upper chord members, in acting as chords in the triangular frames through which the intermediate loads are transferred, are subjected to a stress equal to the horizontal component of the stress in the sub-diagonals, i.e., $-\frac{1}{2}(W'+W_1)\,h'/h \times 2p/h' = -(W'+W_1)p/\text{h}$, which must be combined with the stresses in these members from the loading in Fig. (*b*).

Considering the stresses in the transformed truss, Fig. (*b*), from $\Sigma H = 0$ applied to the forces at A, stress AC is zero. Hence the stress in the corresponding members AB and BC in the actual truss Fig. (*a*), is only that due to their service as part of the triangular frame $Ab'C$, i.e., $-(W'+W_1)p/h$. The load, $W'+\frac{1}{2}W_1$ at A, produces a like compressive stress in Aa, but does not affect the stresses in any other members. In the analysis of these stresses, this load need not be included, therefore, in the reaction R, which may be regarded as the reaction from the loads applied to the truss *between* the end supports. Eliminating AC and Aa, the conditions are as shown in Fig. 2.

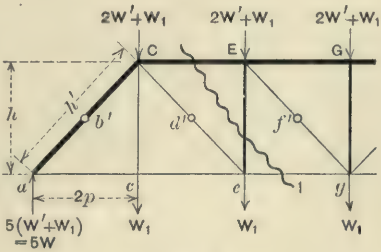


FIG. 2.

The stresses in the diagonals and inclined end post may now be found in the usual way by multiplying the shears in corresponding panels by h'/h. These stresses are the true stresses in the lower halves of these members in the original truss. The stresses in the upper halves are found, as explained above, by combining the stress $+\frac{1}{2}(W'+W_1)h'/h$ with the stresses in the lower halves. Thus, letting $W=W'+W_1$:

$$\begin{array}{ll} ab'=-5Wh'/h & Cb'=-4.5Wh'/h; \\ d'e=+3 \quad `` & Cd'=+3.5 \quad `` \\ f'g=+1 \quad `` & Ef'=+1.5 \quad `` \end{array}$$

The stresses in Cc and Gg are equal to the loads at c and G respectively. The stress in Ee is equal to the load $2W'+W_1$ at E, plus the vertical component, $W'+W_1$, of the stress in Eg. Hence

$$Cc=+W_1; \qquad Ee=-(2W+W'); \qquad Gg=-(W+W').$$

(*b*) *Chord Stresses.* The chord stresses may be found directly from the stresses in the diagonals by applying $\Sigma H=0$ to successive joints, observing that the horizontal component of the stress in any diagonal is the product of that stress by $2p/h'$. Thus, for the lower chord stresses:

$$ac=ce=5Wh'/h\times 2p/h'=(+)10Wp/h,$$
$$eg=(5+3)Wh'/h\times 2p/h'=(+)16Wp/h.$$

The upper chord stresses from Fig. 2 must be increased by $(W'+W_1)p/h=Wp/h$, as explained above. Thus

$$CD=DE=(5+3=8)Wh'/h\times 2p/h'+Wp/h=(-)17Wp/h,$$
$$EF=FG=(8+1=9)Wh'/h\times 2p/h'+Wp/h=(-)19Wp/h.$$

Applying $\Sigma H=0$ to the forces to either side of section 1, it is seen that in Fig. 2 stresses CE and eg are equal, and of opposite sign. Hence stress $CD(=DE)$, Fig. 1 (*a*), might have been found by adding Wp/h to stress eg, i.e.,

$$CD=DE=16Wp/h+Wp/h=(-)17Wp/h,$$

as before.

Second Method. The relation between the primary and secondary web system having been established as above, the stresses may be most readily found directly from Fig. 1 (*a*). The load $\frac{1}{2}W'$ at A may be neglected except for the stress in Aa. The reaction R from the remaining loads is $5.5W$.

(*a*) *Web Stresses.* The dead-load stress in the counter ef' is zero. Since each of the sections 1, 2 and 3 intersects only three members, the stresses in the diagonals cut are the products of the shears in corresponding panels by h'/h. Hence

$$\begin{aligned} Cb' &= (5.5-1)Wh'/h = (-)4.5\ Wh'/h;\\ Cd' &= (5.5-2)\quad \text{``}\quad = (+)3.5\quad \text{``}\\ Ef' &= (5.5-4)\quad \text{``}\quad = (+)1.5\quad \text{``} \end{aligned}$$

As has been seen, the vertical components of the stresses in the upper halves of the tension diagonals exceed those in the lower halves by $\frac{1}{2}W$. Therefore

$$d'e = +3Wh'/h \quad \text{and } f'g = +Wh'/h.$$

The vertical component of stress ab' may be found from $\Sigma V=0$ applied to b'. The load W at that joint and the vertical component $4.5W$, of stress Cb' are exerted downward and the vertical component, $0.5W$, of Ab' is exerted upward. Their downward resultant is therefore $5W$, whence

$$ab' = -5Wh'/h.$$

The same result may be reached by considering that one-half of the load W at b' is transferred to A and transmitted through Aa to a. Combining this with the reaction, $5.5W$, the vertical component of stress ab' is $5W$, as before.

(*b*) *Chord Stresses.* The chord stresses may be derived directly by the method of moments. From section 2, the moment center of ae lies at C and that of CE at e. Similarly, from section 3, the moment center of eg lies at E, and that of EG at g. From moments about these points:

$$\begin{aligned} ac = ce &= (5.5W\times 2p - W\times p) \div h = (+)10Wp/h;\\ eg &= (5.5W\times 4p - 3W\times 2p) \div h = (+)16\quad \text{``}\\ CD = DE &= (5.5W\times 4p - 2W\times 2.5p) \div h = (-)17\quad \text{``}\\ EF = FG &= (5.5W\times 6p - 4W\times 3.5p) \div h = (-)19\quad \text{``} \end{aligned}$$

158. Live-load Stresses from Uniform Loading by the Conventional Method. The live-load stresses from uniform loading in trusses of this type are commonly analyzed by the conventional method (Art. 98) by which full panel loads are assumed to be applied at such joints as may be necessary for maximum stress in any given member.

(*a*) *Chord Stresses.* The chord stresses become maximum throughout when the uniform load extends over the entire bridge. These stresses may therefore be found by multiplying the dead-load stresses by wp/W, in which w is the intensity of the live load per truss, p the panel length, and W the dead load per panel for one truss.

(*b*) *Secondary Members.* For a deck truss, Fig. 3, the live-load stresses in the lower sub-verticals and in hanger Cc are evidently zero. The live-loads tresses in the sub-struts Bb', Dd', Ff' and in the vertical end post Aa', are $-wp$, and in the sub-diagonals, Ab', Ed' and Gf', $+\frac{1}{2}wph'/h$.

(*c*) *Inclined End Post.* The stress in the lower half, ab', is evidently maximum when the whole truss is loaded. Its value, found in the same way as for the dead-load stress, is therefore $-5wph'/h$. For maximum compression in the upper half, Cb', full panel loads must be applied at all upper chord joints from the right end to C. A load wp at B would produce a negative shear $(-\frac{1}{12}wp)$ in panel BC, and therefore a tensile stress of $\frac{1}{12}wph'/h$ in Cb'. This stress is much smaller than the dead-load compressive stress in that member and need not be considered unless the members are to be proportioned by the maximum and minimum method (Art. 101). The desired maximum stress in Cb', as found in the usual manner from the maximum positive shear, $(\frac{1}{12}+\frac{2}{12}+\ldots+\frac{10}{12})wp=\frac{55}{12}wp$, is then $(-)\ \frac{55}{12}\ wph'/h$.

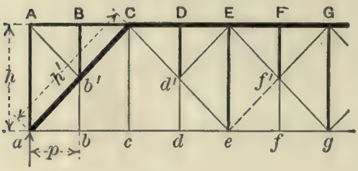


FIG. 3.

(*d*) *Main Diagonals.* For maximum tension in Cd' and Ef', Fig. 1 (*a*), the load must extend from the right end to D and F, respectively, whence

$$Cd'=+\frac{45}{12}wph'/h \quad \text{and} \quad Ef'=\frac{28}{12}wph'/h.$$

Considering the lower half $d'e$ of diagonal Ce, for example, it is evident that loads from E to the right produce tension, and loads from C to the left compression in this member. A load wp at D, Fig. 4, produces a negative shear of $\frac{3}{12}wp$ in panel DE and a tensile stress in $d'E$ whose vertical component is $\frac{1}{2}wp$.

From $\Sigma V=0$ the vertical component of stress $d'e$ from this load is therefore $(\frac{9}{12}+\frac{1}{2}-1)wp=\frac{3}{12}wp$, and its direction is as shown in the figure. A load wp at D thus produces a tensile stress of $\frac{3}{12}wph'/h$ in $d'e$. This must be added to the tensile stress from the loads from E to the right, whence

$$d'e=\left[\frac{3}{12}wp+\left(\frac{1}{12}+\frac{2}{12}+\ldots+\frac{8}{12}\right)wp\right]h'/h=(+)\frac{39}{12}wph'/h.$$

The same result may be reached by finding the total shear, $R-wp$, in panel DE from loads from D to the right. The vertical component of stress $d'e$ is then equal to this shear plus the vertical component $\frac{1}{2}wp$ of stress $d'E$. Thus

$$d'e=\left[\left(\frac{1}{12}+\frac{2}{12}+\ldots+\frac{9}{12}\right)wp-wp+\tfrac{1}{2}wp\right]h'/h=(+)\frac{39}{12}wph'/h.$$

Similarly for maximum tension in $f'g$, the load must extend from F to the right, whence,

$$f'g=\left[\left(\frac{1}{12}+\frac{2}{12}+\ldots+\frac{7}{12}\right)wp-wp+\tfrac{1}{2}wp\right]h'/h=(+)\frac{22}{12}wph'/h.$$

The minimum stresses in the main diagonals may be found by assuming those joints loaded which are free from load for maximum stresses. The resulting live-load compressive stresses must be combined with the dead-load tensile stresses. The minimum stress in $f'g$ is zero. The minimum stress in Ef' is the dead-load stress, $+\frac{1}{2}Wh'/h$, in this member from the dead loads at F and f when the stress in $f'g$ is zero. The minimum stress in Gf' is its dead-load stress, $+\frac{1}{2}Wh'/h$.

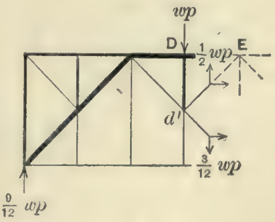


FIG. 4.

(*e*) *Counter Stresses.* Assume first that the counter $f'e$, Fig. 5, does not exist, and let it be required to find the maximum compressive counter stress in $f'g$. A load wp at F will produce a positive shear of $\frac{5}{12}wp$ in panel GF, and a stress in Gf' whose vertical component, $\frac{1}{2}wp$, is exerted downward with respect to G. Taking section 1, and applying $\Sigma V=0$ to the forces to the left of the section, the vertical component of stress $f'g$ is $(\frac{1}{2}-\frac{5}{12})wp=\frac{1}{12}wp$, and it is exerted upward with respect to g, thus indicating that a load at F produces tension in $f'g$. For maximum compression in $f'g$, loads wp must be placed at joints B to E inclusive. The corresponding compressive stress in $f'g$ is $(\frac{1}{12}+\frac{2}{12}+\frac{3}{12}+\frac{4}{12})wph'/h=\frac{10}{12}wph'/h$. If D is the dead-load tensile stress in $f'g$, then by Art. 102 (second method) the resultant compressive stress in $f'g$ is $\frac{10}{12}wph'/h-\frac{1}{2}D$, in which $D=Wh'/h$ (Art. 157). Since $f'g$ is a flexible member it is incapable of developing a compressive stress; and for stability Gf' must act as a counter in tension after the dead-load tension in $f'g$ is neutralized. When that occurs, $f'g$ may be considered removed from the truss. For maximum stress Gf', joints B to F, inclusive, must be loaded, the resulting live-load stress being

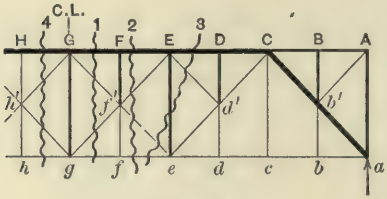


FIG. 5.

$$\left(\frac{1}{12}+\frac{2}{12}+\ldots+\frac{5}{12}\right)wph'/h=(+)\frac{15}{12}wph'/h.$$

The dead-load shear in panel FG is $-\frac{1}{2}W$. When $f'g$ is not acting the dead-load stress in Gf' is therefore $-\frac{1}{2}Wh'/h$. Combining this with the live-load stress, by Art. 102 (second method):

$$Gf'=+\left(\frac{15}{12}wp-\frac{1}{4}W\right)h'/h.$$

Counter ef' must come into play for stability when the dead-load tension in $f'g$ has been neutralized. A load wp at F will produce a compressive stress of $\frac{1}{12}wph'/h$ in ef', as may be seen by taking section 2 and analyzing the stresses as in Fig. 4. The maximum stress in ef' occurs when joints B to E, inclusive, are loaded. Since no live load is at F, the live-load stress in Ef' is zero. The live-load stress in ef' may therefore be found directly from the shear, $+\frac{10}{12}wp$, in panel EF. The resulting stress, $+\frac{10}{12}wph'/h$, must be combined with the dead-load stress which exists in ef' when $f'g$ is not acting. The dead-load shear in panel ef is $-1\frac{1}{2}W$. The vertical component of the dead-load tensile stress in Ef' is $\frac{1}{2}W$, and it is exerted upward with respect to f'. Therefore, from $\Sigma V=0$ applied to the forces to the left of section 2, the dead-load stress in ef' is $-(1\frac{1}{2}W-\frac{1}{2}W)h'/h=-Wh'/h$. Hence by Art. 102 (second method):

$$ef'=+\left(\frac{10}{12}wp-\frac{1}{2}W\right)h'/h.$$

(*f*) *Posts.* When stress $d'e$ is maximum, the counter ef' is not in action. Hence from $\Sigma V=0$ applied to the forces at e, stress Ee is equal to the vertical component of stress $d'e$—i.e., the stresses in these two members become maximum concurrently. From the above value of the maximum stress in $d'e$,

$$Ee=-\frac{39}{12}wp.$$

For minimum stress in Ee, observe that loads wp at B and C, Fig. 5, produce a tensile stress in Ee equal to the reaction, $\frac{3}{12}wp$, at the left support, as may be seen by taking section 3 and considering the forces to the left. The minimum stress may be found by combining this stress with the dead-load compressive stress.

For maximum live-load stress in Gg two cases must be considered.

First, let loads wp be assumed applied at the center G, and at joints F and H on each side of the center. The resulting live-load stress in Gg is $-2wp$, which must be combined with the dead-load stress, $-(W+W')$, from Art. 157. Then by Art. 102 (second method):

$$Gg=-[2wp+\tfrac{1}{2}(W+W')].$$

Second, consider the loads wp applied for maximum stress in $f'g$, i.e., from F to the left. From the above value of maximum stress in $f'g$ its vertical component is $\frac{22}{12}wp$. From $\Sigma V=0$ applied to the forces at g, the live-load stress in Gg equals the resultant of the vertical components of stresses $f'g$ and gh'. The vertical component of stress gh' from the loading under consideration will now be found. The forces to the right of section 4 consist of the reaction $\frac{28}{12}wp$ at a and the loads wp at G and F. The shear in panel GH is therefore $-\frac{4}{12}wp$. The vertical component of stress Gh' is $\frac{6}{12}wp$, and is exerted downward with respect to G. From $\Sigma V=0$ applied to the forces to the right of section 4, the vertical component of stress gh' is $\frac{2}{12}wp$, which acts upward with respect to g. It is seen, therefore, that under the assumed loading the diagonals gh' and $f'g$ are both subjected to tension from the live load. The corresponding live-load stress in Gg is $\frac{22}{12}wp+\frac{2}{12}wp=(-)2wp$. This stress is seen to be equal to that for the condition of loading first assumed. The dead-load stress in Gg is also the same in both cases, since the diagonals at g are both in action, for both conditions of loading.

For the first condition of loading, the stress in the central post is evidently unaffected by the number of panels in the truss; and for a 12-panel truss the resulting stress was found to be the same as for the second condition of loading. A little consideration will show that if the number of panels is 8, the first case governs, and that if the number of panels is 16 the second case may govern. When the diagonals which intersect at the foot of the central post are not both in action under the assumed loading, as may be true for a 16-panel truss under the second condition of loading. the dead-load stress in the post must be computed for the transformed truss and then combined with the live-load stress.

For minimum stress in Gg, Fig. 5, observe that the dead-load tensile stresses in the sub-diagonals Gh' and Gf' cannot be reduced by live-load compressive stresses. From $\Sigma V=0$ at G, the minimum stress in Gg is therefore its dead-load stress, $-(W+W')$.

159. Live-load Stresses from Concentrated Loading. The criteria for determining the position of a concentrated load system for maximum stress in any given member will be deduced later (Arts. 165 and 166) from those applicable to the general case of trusses with inclined chords. After the position of the loading is known the corresponding web stresses may be found from the shears, and the chord stresses by moments.

160. Transmission of Loads by the Secondary Web Systems. Referring to Fig. 6 (a), the panel load W at any intermediate joint, as d, for example, may be transmitted to the primary web system either at c and e, Fig. (b), through the triangular frame $cd'e$, or at C and c, Fig. (c), through the triangular frame $Cd'c$. The former assumption is analogous to that made in the preceding article. It will be shown now that the resulting stresses will be alike by either assumption. Since the conditions at joint c are the same in Figs. (b) and (c), the hanger Cc is not involved. The only stresses to be considered are those in the diagonals Cd' and $d'e$ and in the chord members CE and ce.

In Fig. (*b*) the center of gravity of the two component loads $\frac{1}{2}W$ lies in the same vertical line with the point of application, d, of the original load, W, and the two horizontal components, H, are balanced. In Fig. (*c*) the center of gravity of the component loads lies at a distance p to the left of the original load. The reaction at a from the vertical loading is therefore increased an amount Wp/l, and the other reaction is decreased by a like amount. This effect is neutralized, however, by the couple formed by the horizontal components H, whose moment Hh is equal to the moment Wp, but opposite in direction. This condition must be fulfilled, for since the four component forces H and $\frac{1}{2}W$ in Fig. (*c*) are the true components of W, they must be held in equilibrium by the same end reactions which held W, their resultant, in equilibrium.

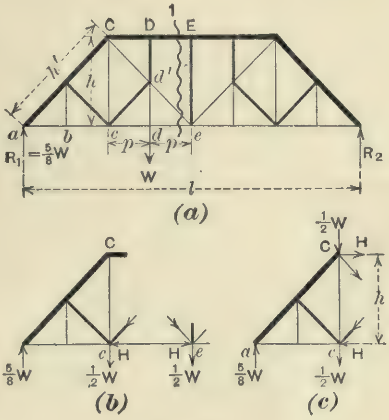


FIG. 6.

Referring to Fig. (*a*), the shear in panel de is $\frac{5}{8}W - W = -\frac{3}{8}W$. Hence stress $d'e$ is $-\frac{3}{8}Wh'/h$. If the load W is transmitted, as in Fig. (*b*), the stress incident to this transfer is $-\frac{1}{2}Wh'/h$ in $d'e$ and zero in Cd'. The stress in both members from the redistributed loading is $+\frac{1}{8}Wh'/h$, and the stresses from W in its original position are therefore

$$Cd' = +\tfrac{1}{8}Wh'/h \qquad \text{and} \qquad d'e = (\tfrac{1}{8} - \tfrac{1}{2})Wh'/h = -\tfrac{3}{8}Wh'/h.$$

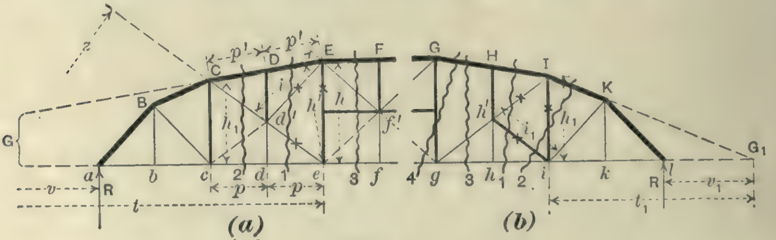
If the load W is transmitted as in Fig. (*c*), the stress incident to this transfer is $+\frac{1}{2}Wh'/h$ in Cd' and zero in $d'e$. The stress in both members from the redistributed loading is $-\frac{3}{8}Wh'/h$, which is also the stress in $d'e$ from W in its original position. For Cd' this stress is $(\frac{1}{2} - \frac{3}{8})Wh'/h = +\frac{1}{8}Wh'/h$. Thus the stresses in the diagonals are seen to be the same by either assumption as to the transmission of W.

Reverting to Fig. (*a*), the stress in CE may be found by moments about e, i.e., stress $CE=[(\frac{5}{8}W\times 4p)-(W\times p)]\div h=(-)\frac{3}{2}Wp/h$. If the load is transmitted as in Fig. (*b*), the stress in CE incident to this transfer is zero. This is also seen by taking moments about e, whence stress $CE=[(\frac{5}{8}W\times 4p)-(\frac{1}{2}W\times 2p)]\div h=(-)\frac{3}{2}Wp/h$, as before. In Fig. (*c*), however, the redistribution of the load induces a stress in CE of $H=-Wp/h$, which must be combined with the stress in CE from the loading in that figure. The latter stress is $[(\frac{5}{8}W\times 4p)-(W\times 2p)]\div h=(-)\frac{1}{2}Wp/h$, and the total stress in CE is therefore $(-)\frac{3}{2}Wp/h$, as before.

Since it has been shown now that the stresses in the diagonals and in the upper chord are both unaffected by the assumption as to the transmission of W, it follows that this is true also of the lower chord stress ce. For, taking section 1, from $\Sigma H=0$ applied to the forces on either side of the section, the algebraic sum of chord stresses DE and de and the horizontal component of stress $d'e$ must equal zero.

161. Pratt Truss with Inclined Chords and Sub-Panels (Pettit Truss). This type of truss, Fig. 7, is used more extensively than any other for spans above 300 ft. for railroad bridges and 225 ft. for highway bridges. It will therefore be fully analyzed for concentrated as well as uniform live loading, and for the two different arrangements of the secondary web system, shown in Figs. (*a*) and (*b*).

162. Dead-load Stresses. Let W' and W_1 represent, respectively, the upper and lower panel concentrations, and let $W'+W_1=W$. Then for the truss shown in Fig. 7, $R=4.5W$.



(*a*) (*b*)

Fig. 7.

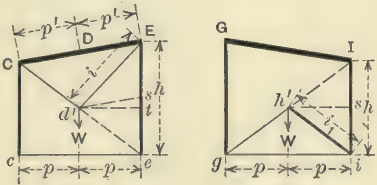
Web Stresses. The stresses in the upper and lower subverticals are $-W'$ and $+W_1$, respectively. The stresses in aB, Bc, Cc, and Cd', Fig. (*a*), and in lK, Ki, $h'g$, and Gg, Fig. (*b*), are found in precisely the same manner as in a truss without sub-panels (Art. 132). The stress in Gf' is found, as in Art. 157, i.e., its vertical component is $\frac{1}{2}W$. The only remaining web members which require further consideration are those distinguished by cross-marks, forming the triangles $d'eE$ and $h'iI$ in Figs. (*a*) and (*b*) respectively.

The stresses in the sub-diagonals are

$$Ed' = Wi/h \quad \text{and} \quad h'i = Wi_1/h_1, \quad . \quad . \quad . \quad (1)$$

which may be proved as follows:

The double panels ce and gi, Fig. 7, are shown separately in Fig. 8. The load W conveyed to d' by the sub-verticals is assumed to be transmitted thence by the diagonals represented by full lines. For present purposes the members $d'e$, Fig. (*a*), and Ih', Fig. (*b*), may be considered removed from the truss.



(*a*) (*b*)

FIG. 8.

In Fig. (*a*), the triangle $Ed'e$ may be regarded as a force triangle of the three forces meeting at d'. Therefore to the same scale to which Ee represents W, Ed' and $d'e$ represent the stresses from W in Ed' and Cd', respectively. Thus the desired stress $Ed' = (+)W/h \times i$; and similarly in Fig. (*b*) stress $h'i = (-)W/h_1 \times i_1$, as in Eq. (1).

Again, in Fig. (*a*) since d' bisects Ce, $d's$ drawn parallel to CE bisects Ee. Regarding the triangular frame CEd' as an independent truss carrying a central load W, the triangle $Ed's$ may be considered a force triangle of the forces at E, in which $Es (= \frac{1}{2}W)$ represents the reaction, and $d's$ $(= W/h \times p')$ represents stress CE. It is also seen that Es is the difference between the vertical components Et of stress Ed' and st of stress CE, as it should be.

Similarly, in Fig. (*b*), drawing $h's$ parallel to gi, the triangle $h'is$ may be considered a force triangle of the forces at i, in which $is (= \frac{1}{2}W)$ represents the reaction, and $h's (= W/h_1 \times p)$ represents stress gi.

In Fig. (a) $d't$ represents the horizontal component of stress Ed'. Since $d't = W/h \times p$, it is seen that this component is not affected by the inclination of the chord, and that its value is constant if h is constant. The horizontal components of the stresses in the diagonals on opposite sides of the same post, from the intermediate panel loads W are therefore equal and opposite. This relation is useful in the analysis of chord stresses, as will be seen below.

Reverting now to Fig. 7, the transfer of W in equal parts to C and E, Fig. (a), induces no stress in $d'e$, and Ed' may then be regarded as removed from the truss. Section 1 may now be taken without severing more than three members, and stress $d'e$ may be found in the same manner as for a truss without sub-panels.

Thus by Eq. (13), Art. 132, and with the notation shown in Fig. 7 (a), expressing all lever arms in terms of panel lengths,

$$d'e = [4.5Wv - W(v+1) - 1.5W(v+2)] \div z = (+)(2v-4)W/z,$$

or by Eq. (16), Art 132,

$$d'e = [2Wv - (W + 1.5W \times 2)] \div z = (+)(2v-4)W/z.$$

Stress Ih', Fig. 7 (b), may be found in like manner, by considering the load W brought to h' transferred in equal parts to g and i, and then taking section 1. The above equations for $d'e$ will be found to apply without change to Ih', i.e., stress Ih' = stress $d'e$.

From Eqs. (14) and (17), Art. 132, in their respective relations to Eqs. (13) and (16), it is seen that stress Ee, Fig. (a), may be derived from stress $d'e$ as follows:

$$Ee = (d'e)z/t - W_1 = (-)[(2v-4)W/t - W_1].$$

For stress Ii, Fig. (b), taking section 2, the moment center lies at G_1. Stress Ii may be found by Eq. (14) or (17), Art. 132, noting, however, that the load W_1 at i must be increased by $\frac{1}{2}W$, which is assumed to have been carried from h' to that point before section 4 was taken. By Eq. (14),

$$Ii = \frac{4.5Wv_1 - W(v_1+1)}{t_1} - (W_1 + \tfrac{1}{2}W),$$

whence

$$Ii=(-)[(3.5v_1-1)W/t-(W_1+\tfrac{1}{2}W)].$$

Chord Stresses. The chord stresses may be conveniently found by moments in the same manner as for trusses with horizontal chords (Art. 157, second method) except that the lever arm by which the resultant moment is divided is affected by the inclination of the chord. Thus in Fig. 7 (*a*), taking section 2, the moment center of *ce* lies at *C*, and that of *CE*, at *e*. From moments about these points of the forces to the left of section 2:

$$ce=[(4.5W\times 2p)-(2W\times 0.5p)]\div h_1=(+)8Wp/h_1,$$

$$CE=[(4.5W\times 4p)-(2W\times 2.5p)]\div h'=(-)13Wp/h',$$

in which $h'=hp/p'$. Substituting this value of h' in the equation for *CE*,

$$CE=(-)13Wp\div hp/p'=(-)13Wp'/h.$$

If the sub-diagonals are arranged to act in compression instead of tension, the tie *d'E* is omitted and the strut *h'i*, Fig. (*b*), is added. Section 3 must then be taken in the next panel from the end, in order that it may cut only three members. The moment centers of *gi* and *GI* lie at *I* and *g*, respectively, and the stresses have now the following values:

$$gi=[(4.5W\times 2p)-(3W\times 0)]\div h_1=(+)9Wp/h_1,$$

$$GI=[(4.5W\times 4p)-(3W\times 2p)]\div h'=(-)12Wp/h'.$$

Reverting to Fig. (*a*), the dead-load stress in counter *ef'* is zero. Taking section 3, the moment centers of *eg* and *EG* lie at *E* and *g* respectively. Hence

$$eg=[(4.5W\times 4p)-(4W\times 1.5p)]\div h=(+)12Wp/h,$$

$$EG=[(4.5W\times 6p)-(4W\times 3.5p)]\div h=(-)13Wp/h.$$

It is seen that stress *EG* is equal to stress *eg* plus horizontal component, Wp/h, of stress *Ef'*, as it should be.

Stresses *EG* and *eg* might have been derived directly from the chord stresses previously obtained, in the following manner:

Applying $\Sigma H=0$ to the forces at *E*, Fig. (*a*), the horizontal components of *Ed′* and *Ef′* are equal and act in opposite directions, as was seen above in the analysis of web stresses. The horizontal components of the chord stresses *CE* and *EG* are therefore also equal and opposite, i.e., $EG=CE\times p/p'=CE\times h'/h$. Again, taking section 4 in Fig. (*b*), and applying $\Sigma H=0$ to the forces on either side of the section, stress *eg* is equal to the horizontal component of stress *GI*, i.e., $eg=GI\times p/p'=GI\times h'/h$.

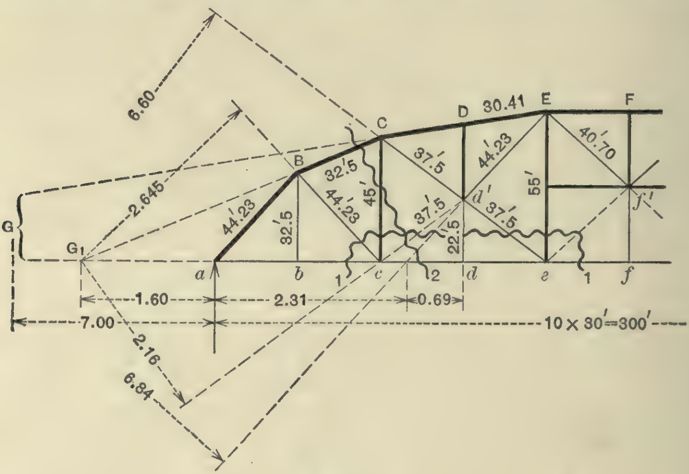


FIG. 9.

163. Example. Dead-load Stresses in a Pettit Truss. Required the dead-load stresses in the 300-ft. single track railroad bridge shown in Fig. 9, from the following loads per foot of bridge:

Steel	Trusses and Bracing,	2700	lbs.
	Floor beams and Stringers,	700	"
	Total,	3400	"
	Track,	500	"
	Grand total,	3900	"

Solution. All loads and stresses will be expressed in thousands of pounds. The weights of the steel floor system and track will be assumed to be applied wholly to the lower chords. The remaining weight will be equally divided between the four chords.

Apex Loads:

$$\text{Lower,}\quad \left(\frac{0.7+0.5}{2}+\frac{2.7}{4}\right)30=38.25$$

$$\text{Upper,}\quad \frac{2.7}{4}\times 30 \qquad =20.25$$

$$\text{Total,}\qquad =58.50$$

Check: $\frac{3.9}{2}\times 30=58.5.$

$$\text{End Reaction}=58.5\times\frac{9}{2}=263.3.$$

(*a*) *Numeric Method.* Although, as pointed out in the preceding, article, the derivations of many of the stresses involve no new principles, the stresses will all be computed, so that the comparison between this method and the graphic method below may be completely indicated.

Lever Arms in Terms of Panel Lengths (Art. 131):

$$\frac{32.5}{45-32.5}-1=1.60; \qquad \frac{45}{50-45}-2=7.00$$

$$\frac{22.5}{55-22.5}=0.69.$$

$$Bc,\quad (1.6+2)\frac{32.5}{44.23}=2.645; \qquad Ce,\quad (7+4)\frac{45}{75}=6.60;$$

$$cd',\quad (1.6+2)\frac{22.5}{37.50}=2.16; \qquad Ed',\quad (7+2.31)\frac{32.5}{44.23}=6.84$$

End Post:

$$aB=263.3\times\frac{44.23}{32.5}=(-)358.3.$$

Main Diagonals: By Eq. (16), Art. 132,

$$Bc\ =[58.5(\overline{3.5\times 1.6}-1)=269.1]\div 2.645=(+)101.7$$

$$Cd'=[58.5(\overline{2.5\times 7\ \ }-3)=848.3]\div 6.6 \quad =(+)128.5$$

$$d'e\ =\{58.5[2\times 7-(1+\overline{1\tfrac{1}{2}\times 2)}]=585.0\}\div 6.6=(+)88.6$$

Sub-diagonals:

$$d'E=58.5\times\frac{44.23}{55}=(+)47.0; \qquad Ef'=58.5\times\frac{40.70}{55}=(+)43.3$$

Counters. When the dead-load stress in *d'e* has been neutralized by the live-load counter stress, that member may be considered as removed from the truss, and $d'E$ will come into play as a counter. Its dead-load stress is then

$$d'E=[58.5(\overline{1.5\times 7}-6)=263.3]\div 6.84=(-)38.5.$$

When counter cd' acts, its stress is equal in magnitude but opposite in sign to the stress which ed' would have under the same loading, if the counter did not exist. This may be seen by taking a section 1–1 separating joints c, d and e from the remainder of the truss, and applying $\Sigma H=0$ with first cd' and then $d'e$ acting, noting that the change does not affect the stresses in any members outside the double panel ce. Therefore

$$cd'=-d'e=-88.6.$$

Verticals.

$$Bb=dd'=ff'=+38.3;$$

$$Dd'=Ff'=20.3;$$

$$Cc=269.1\,(\text{see } Bc)\div 3.6-38.25=(-)36.5;$$

$$Ee=585\ (\text{see } d'e)\div 11-38.25=(-)14.9.$$

The dead-load stress in Cc when the counter cd' is in action under a compressive dead-load stress of 88.6, as found above, may be obtained by taking section 2, and considering moments about G_1. Thus

$$Cc=[(263.3\times 1.6)-(58.5\times 2.6)-(38.25\times 3.6)-(88.6\times 2.16)]\div 3.6=(+)16.6$$

Chords.

$$ab(=bc)=263.3\times\frac{30}{32.5}=(+)243.0.$$

By Art. 162.

$$cd\ (=de)=[(263.3\times 2)-(2\times 58.5\times 0.5)]\frac{30}{45}=(+)312.1;$$

$$ef\ =[(263.3\times 4)-(4\times 58.5\times 1.5)]\frac{30}{55}=(+)383.0;$$

$$BC\ =312.1\,(\text{see } cd)\times\frac{32.5}{30}=(-)338.0;$$

$$CD(=DE)=[(263.3\times 4)-(2\times 58.5\times 2.5)]\frac{30.41}{55}=(-)420.7$$

$$EF\ =420.7\,(\text{see } CD)\times\frac{30}{30.41}=(-)415.0.$$

(*b*) *Graphic Method.* The dead-load stresses may be conveniently found graphically, as shown in Fig. 10. The full lines in Fig. (*b*) represent the stresses when the counter OP, Fig. (*a*), is not in action. These stresses are found in the usual way. Beginning at joint 1, the reaction $R(=263.3)$ is laid off as ka, and after constructing the force polygon for the forces at that joint, the same procedure is applied to the other joints in the sequence of the numerals at these joints.

When the tensile dead-load stress in *QT* is neutralized by the compressive live-load stress, that member may be considered removed from the truss, and *ST* comes into play as a counter. The counter *OP* is then subjected to compression from the dead load, and the stress in every member within the quadrangle 4–6–10–9 (except the sub-verticals) including its sides, is affected. The resulting changes are indicated by broken lines and primed letters in Fig. (*b*). In constructing this modified diagram, begin at joint 9, remembering that stress *QT* is zero, and that stress *UF* is unaffected. The force polygon is *ufgq′t′u*. Proceed then to joint 10, at which the stresses in *DS* and *ST* are unknown, or to joint 4, at which the stresses in *NO* and *OP* are unknown. After the force polygon has been drawn for either of these joints, the remaining two joints of the quadrangle may be considered in either order.

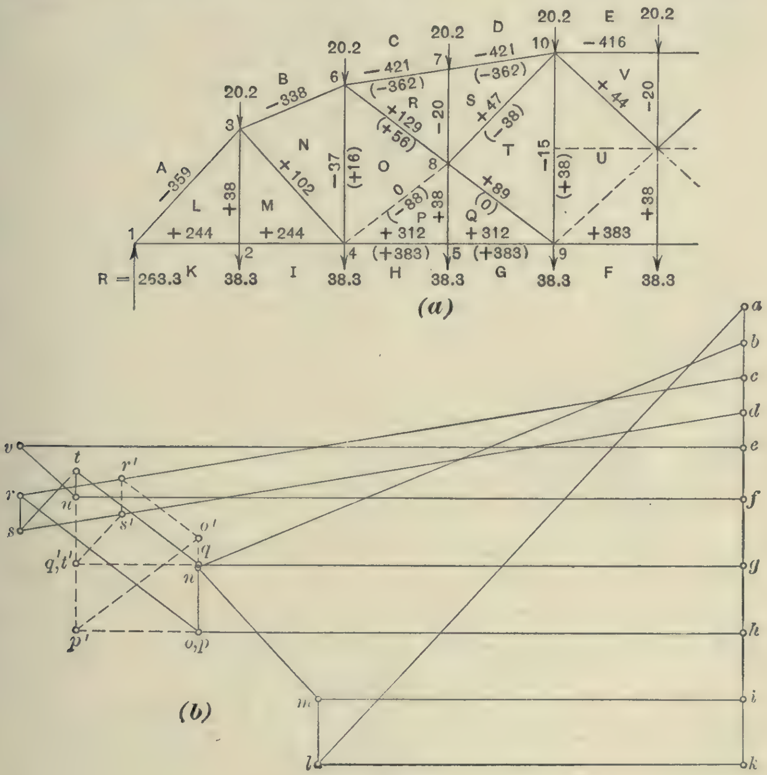


FIG. 10.

The modified stresses when *ST* acts as a counter are enclosed in parentheses in Fig. (*a*). It is seen that the stresses in the posts *NO* and *TU* are now tensile, the stress in *TU* being equal to the load 38.3 at joint *g*. The stress in the sub-diagonal *ST* is also reversed in sign. The stresses in the upper chord members *CR* and *DS*, and in diagonal *OR* are smaller than before. These stresses are of interest only if the members are to be proportioned by the method involving the minimum as well as the maximum stress in each member (Art. 101). In that case they must be combined with the stresses from the minimum live load which will bring the counter into action. For each member the combined stress or the normal dead-load stress is to be regarded as the "minimum" stress, according to which is lower. The lower chord stresses $HP = GQ$ are greater than before, but since the counter *OP* is not in action when the live load extends over the entire bridge (for maximum live-load chord stresses) these stresses are of no practical interest.

For reasons explained above in connection with the numeric method, stresses *OP* and *QT* should be equal in magnitude and opposite in sign, whereas the values found for these stresses are $+89$ and -88 respectively, the discrepancy being due to inaccuracies in drawing and scaling. The computed value is 88.6. Comparing the stresses obtained by the graphic method with those previously found by computation, it will be seen that the differences range from zero to less than one per cent.

164. Live-load Stresses from Uniform Load. Conventional Method. Referring to Fig. 11, let w equal the intensity of the uniform load per truss, and p the panel length. It is required to determine (1) the joints at which full panel loads wp must be applied for maximum stress in any given member, and (2) the values of these stresses.

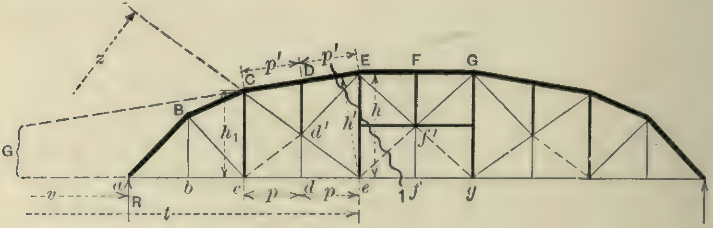


FIG. 11.

(*a*) *Chord Stresses.* The chord stresses will be maximum throughout the truss when the load extends over the entire bridge. The resulting stresses in the chords may therefore be found in precisely the same manner as the dead-load chord stresses in the two preceding articles.

(*b*) *Web Stresses.* Stresses aB, Bc, Cc and Cd' may be found in the same manner as in a truss without sub-panels (Art. 134). For maximum stress in Cc and Cd' assume a load wp applied at every joint from d to the right.

The upper sub-verticals can receive no stress from the live loading. The maximum stresses in the lower sub-verticals, and in Bb, are wp.

Diagonals. The stresses in the diagonals in the middle double-panel eg, in which both chords are horizontal, are found as in Art. 158. Thus for maximum stress in counter $f'g$ the joints from g to the right must be loaded. Then

$$f'g=\frac{1}{10}wp(1+2+3+4)h'/h=(+)wph'/h. \quad . \quad . \quad (a)$$

For maximum stress Ef', another load wp must be applied at f. Then

$$Ef'=\frac{1}{10}wp(1+2+\ldots+5)h'/h=(+)1.5wph'/h. \quad . \quad (b)$$

Considering $d'e$, loads from e to the right produce no stress in the sub-diagonal Ed'. For such loading that member may be considered removed and the stress in $d'e$ may then be found in the same manner as in a truss without sub-panels. Thus,

$$d'e=\frac{1}{10}wp(1+2+\ldots+6)v/z=(+)2.1wpv/z. \quad . \quad (c)$$

A load wp at d may be assumed transferred in equal parts to C and E. The reaction at a from such a load is $\frac{7}{10}wp$. The resulting stress in $d'e$ is $(\frac{7}{10}wp\times v-\frac{1}{2}wp\times\overline{v+2})\div z$, the lever arms being expressed in terms of panel lengths. If $\frac{7}{10}wp\times v>\frac{1}{2}wp\times(v+2)$ this stress is tensile and should then be added to the stress previously found. In that case this load produces compression in Ee, as may be seen from $\Sigma V=0$, applied at e.

For maximum stress in counter $d'c$, joints b and c should be loaded. The analysis of this stress and its combination with the dead-load stress will be fully explained in the next article.

Posts. Loads from f to the right cause compression in Ee. The resulting reaction at a is $1.5wp$, and considering forces to the left of section 1, the corresponding stress is

$$Ee=-1.5wpv/t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

For the loading in this position, counter ef' is not in action. This member may therefore be assumed removed from the truss. Now suppose that additional load wp is placed at e, and let the effect of this load on stress Ee be considered. Since this load produces no stress in Ed', section 1 may be taken, cutting only three active members. From moments about G of R $(=0.6\,wp)$ and wp at e, it is seen that stress Ee from this load is tensile.

Let an added load wp now be placed at d. Assume this load transferred in equal parts to C and E. Then taking section 1, as before, and moments about G of the forces to the right of that section, it is seen that this load produces compression in Ee if $0.7\,wp \times v > 0.5\,wp\,(v+2)$ —i.e., if $v>5$.

It follows, therefore, that if $v<5$, the compressive stress in Ee is maximum when a panel load wp is placed at every joint from f to the right. The stress may then be found by Eq. (d).

If $v>5$ it must be ascertained whether the compressive stress, S_d, in Ee from a load wp at d exceeds the tensile stress, S_e, from a like load at e. Although the load at d produces tension in Ee, that load cannot be assumed omitted, since full panel loads cannot be developed at d and f unless the uniform load covers the entire intervening distance. The values of stresses S_e and S_d are

$$S_e = +\left(1 - 0.6\frac{v}{t}\right) wp,$$

$$S_d = -\frac{0.2v - 1}{t}\,wp.$$

For a ten-panel truss $t=v+4$. Substituting that value for t and comparing these equations, it is found that $S_d > S_e$ when $v < -25$. Since this condition cannot be fulfilled, the compressive stress in Ee will be maximum when a panel load is applied at every joint from f to the right.

It is evident by considering moments about G that added loads wp at c and d will cause tension in Ee.

The live load induces tension in Ee, when the vertical (upward) component of stress CE exceeds the sum of the vertical (downward) components of stresses Ed' and Ef', as is evident from $\Sigma V=0$ applied to the forces meeting at E.

165. Web Stresses from Concentrated Live Loads. (*a*) *Main Diagonals.* Referring to Fig. 12, the conditions for *Bc* and the upper half *Cd′* of diagonal *Ce* are precisely the same as for a truss without sub-panels, the critical weight, for the loading advancing from the right, being placed at *c* and *d*, respectively, to fulfill the criteria in Art. 136. The stresses may then be found as explained in that article.

For *Ef′*, the analysis is the same as for a simple Pratt truss with horizontal chords (Art. 111), the critical weight being placed at *f*.

For maximum stress in *d′e*, the critical load is placed at *e*. Under ordinary conditions no loads will occur to the left of *c* when the criterion for this member, which will now be derived, is satisfied. Let r_c and r_d denote the floor-beam reactions at *c* and *d*, respectively, from the load W_1 in the double panel *ce*, and let it be assumed that the joint load r_d at *d* is transferred in equal parts to *C* and *E*, without affecting the stress in *d′e*.

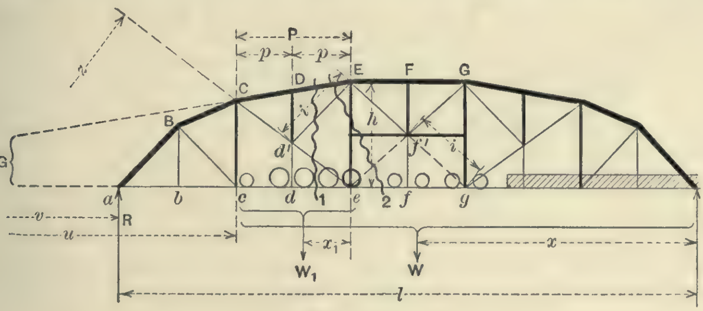


FIG. 12.

The sub-diagonal *Ed′* may then be considered removed from the truss and section 1 may be taken without severing more than three members. The external forces to the left of the section are *R*, r_c, and $\frac{1}{2}r_d$ at *C*, the last two having the same lever arm with respect to the moment center, *G*. The value of $r_c+\frac{1}{2}r_d$ will now be found. The loading W_1 and the reactions at *c*, *d* and *e* from this loading represent four forces in equilibrium. Therefore, taking moments about *e*, $(r_c\times 2p)+(r_d\times p)-W_1x_1=0$, whence

$$r_c+\tfrac{1}{2}r_d=\frac{W_1x_1}{2p}=\frac{W_1x_1}{P}.$$

Now letting M_G denote the moment about G of the forces to the left of section 1,

$$M_G = \frac{Wx}{l}v - \frac{W_1 x_1}{P}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For maximum M_G,

$$\frac{dM_G}{dx} = \frac{Wv}{l} - \frac{W_1 u}{P} = 0,$$

or

$$W = \frac{u}{v}\frac{l}{P}W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the criterion for maximum stress in the lower halves of the main diagonals in panels with inclined chords. Eq. (3) is seen to be precisely the same as Eq. (25) of Art. 136, for a truss with inclined chords, but without sub-panels, except that p, the panel length in Eq. (25), is replaced in Eq. (3) by the double panel length P, and W_1 denotes now the loading in the double panel.

If the truss has N equal double panels, the criterion becomes

$$W = \frac{u}{v}NW_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which u and v may be conveniently expressed in panel lengths.

For parallel chords v = infinity. As v approaches infinity, u/v approaches unity. Therefore, when both chords are horizontal,

$$W = NW_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the criterion for the lower halves of the diagonals in the Baltimore truss (Art. 156), N denoting the number of double panels and W_1 the load in the double panel. This criterion is applicable to member $f'g$, Fig. 12, the critical wheel being placed at g. For Ef' the conditions are the same as for a truss without subpanels, so that in Eq. (5) N denotes then the number of single panels and W_1 the load in the single panel, as in Eq. (40), Art. 111. For Ef' the critical wheel is placed at f.

The position which satisfies the criterion having been found with the aid of the moment table, the corresponding stress in $d'e$ is M_G/z. In Eq. (2), let $R = Wx/l$ and $m_e = W_1 x_1$. The stress may then be expressed as follows:

$$d'e = \frac{Rv - m_e \frac{u}{P}}{z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which R and m_e are readily found with the aid of the moment table. The lever arms u, v and z may be conveniently expressed in panel lengths.

If the sub-diagonals are arranged to act as struts instead of ties, $d'E$ is omitted and cd' added. In that case the foregoing analysis applies without change to the upper half Cd' of diagonal Ce, and the lower half $d'e$ is then in the same condition as in a truss without sub-panels.

(*b*) *Sub-diagonals.* If the dead-load tensile stress in $d'e$ is subject to complete reversal by the live-load compressive stress in that member, a counter cd' (not shown in the figure) must be provided and sub-diagonal Ed' will then act also as a counter. This case will be considered presently. In the absence of counter cd', sub-diagonal Ed' is subject to live-load stress only from the floor-beam load at d. That load, W_l, becomes maximum when the criterion, Eq. (35), Art. 45, is satisfied, and its value may then be found by Eq. (37) or (39) of that article. Then, by Eq. (1), Art. 162,

$$Ed' = W_l \frac{i}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(*c*) *Posts.* With the arrangement of sub-diagonals shown in Fig. 12, the analysis for the posts is the same as for a truss without sub-panels (Art. 136). The moment center of Cc lies at the intersection of BC and cd produced, and the critical weight is placed at d. Again, taking section 2, the moment center of Ee lies at G, and the critical weight is placed at f. The conditions for maximum tensile counter stress in the posts will be considered later (Art. 169).

(*d*) *Counters.* For the counter $f'g$, the critical weight is placed at g, and the criterion is given in Eq. (4). After the position for maximum stress has been determined the corresponding stress in $f'g$ may be found by assuming the panel load r_f at f to be transferred in equal parts to E and G. The stress may then be found in the usual manner from the shear $R-(r_e+\frac{1}{2}r_f)$ in panel fg. If m_g denotes the moment about g of the load in the double panel eg, $r_e+\frac{1}{2}r_f=m_g/P$, whence,

$$f'g=\left(R-\frac{m_g}{P}\right)\frac{i}{\frac{1}{2}h}, \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

in which R and m_g are readily found with the aid of the moment table.

Considering counter $d'c$, Fig. 13, the dead-load tensile stress in ed' is assumed to be neutralized by the live-load compressive stress before the counter comes into play. The stress in the counter is then the combined stress in that member from the dead and live load if the member ed' is considered removed from the truss. In general when counter $d'c$ acts its stress is equal in magnitude but opposite in sign to the stress which ed' would have under the same loading if the counter did not exist. This may be seen by taking a section 1–1, separating joints e, d and c from the remainder of the truss, and applying $\Sigma H=0$ with first ed' and then $d'c$ acting, observing that this change does not affect the stresses in any members outside the double panel ec. Therefore, if $+S_D$ is the dead-load stress in ed', the corresponding stress in the counter, if ed' is considered removed from the truss, is $-S_D$.

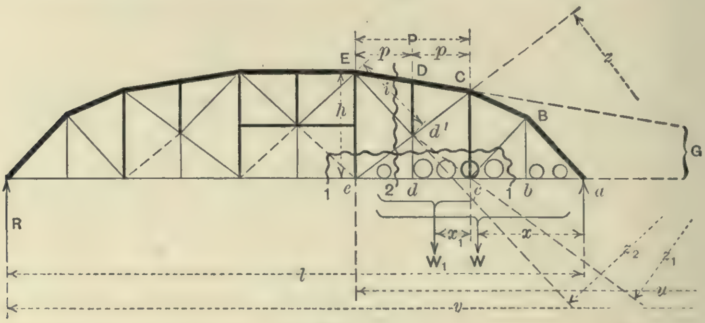


FIG. 13.

In practice it is frequently assumed, by supposed analogy with the conditions for a truss without sub-panels (Art. 134), that a dead-load stress of $+S_D$ in $d'e$ is equivalent to a stress of $-S_D z/z_1$ in cd', if $d'e$ does not exist, whereas the correct value of the equivalent stress is $-S_D$.

It follows directly from the foregoing that the maximum live-load tensile stress in counter $d'c$ is equal in magnitude but opposite in sign to the maximum counter stress which ed' would have if $d'c$ did not exist. Therefore, considering $d'c$ removed from the truss, let the critical weight be applied at c. Assume the load at d transferred in equal parts to E and C without affecting the stress ed'. The sub-diagonal Ed' may then be eliminated, and section 2 may be taken without cutting more than three members. Considering the moment M_G about G of the forces to the left of the section, the sum of the load at e and one-half the load at d, transferred to E, is equal to W_1x_1/P, as explained above in the analysis of stress $d'e$. Thus

$$M_G=\frac{Wx}{l}v-\frac{W_1x_1}{P}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation is seen to be precisely the same as Eq. (2) for the lower half of the main diagonal Ce, Fig. 12. The criteria for the latter, Eqs. (3) and (4), are therefore applicable to counter $d'c$, observing that v denotes, as before, the distance from the left reaction to the moment center, and u the distance from the left end of the double panel P containing the member in question to the moment center.

The position which satisfies the criterion having been found with the aid of the moment table, the corresponding counter stress in ed' if $d'c$ did not exist would be $-M_G/z$. Hence the maximum live-load stress in $d'c$ is $+M_G/z$, which may be combined with the dead-load stress as in Art. 102, second method, whence stress $d'c=M_G/z-\frac{1}{2}S_D$. In Eq. (9) let $R=Wx/l$ and $m_c=W_1x_1$. The stress in the counter may then be expressed as follows:

$$d'c=\frac{Rv-m_c\frac{u}{P}}{z}-\tfrac{1}{2}S_D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which R and m_c are readily found with the aid of the moment table. The lever arms u, v and z may be conveniently expressed in panel lengths.

Considering the counter stress in Ed', that member will come into play as a counter when the dead-load tensile stress in ed' has been neutralized, when ed' may be considered removed, and section 2 will cut only three members. The dead-load stress, $-S_D$, in Ed' may then be found by dividing the moment about G of the forces to the left of the section by the lever arm z_2.

For maximum counter stress in Ed', the critical weight is placed at d. The analysis is precisely the same as for a truss without sub-panels, and the criteria expressed by Eqs. (33) and (34), Art. 136, are applicable to this case without change, observing that W_1 denotes the total load in the single panel de, and p the length of a single panel.

After the position for maximum stress has been determined the stress may be found by taking moments about G of the forces to the left of section 2. If r denotes the forward panel load at e, the live-load stress in Ed' is $(Rv-ru)/z_2$, and combining this with the dead-load stress in the same manner as heretofore,

$$Ed'=\frac{Rv-ru}{z_2}-\tfrac{1}{2}S_D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which R and r may be readily found with the aid of the moment table. The lever arms u, v and z_2 may be conveniently expresssed in panel lengths.

For maximum stress in Ed' a second condition of loading, which sometimes gives a greater result than the foregoing, must be investigated. Let it be assumed that the first or second engine occupies the position for maximum stress in the sub-vertical $d'd$ (Art. 45), and that counter $d'c$ does not come into play for this loading. Then by Eq. (7) above and Eq. (1), Art. 162,

$$Ed'=W_l\frac{i}{h}+W\frac{i}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In proportioning a member for dead and live-load stresses of the same character, as in Eq. (12), these stresses are seldom combined by direct addition as in this equation, but by various methods explained in Art. 102. Eqs. (11) and (12) can only be compared in the light of the particular method to be used in combining dead and live-load stresses. Member Ed' should therefore be proportioned for the stresses by each of the two conditions of loading assumed above, and the larger resulting sectional area should be used.

166. Chord Stresses from Concentrated Live Loads. Referring to Fig. 14, the conditions for the lower chord stresses are precisely the same as for a truss without sub-panels. Thus, considering stress *ce*, for example, and taking section 1, the moment center lies at *C*. The expression for the moment about that point of the forces to the left will involve only the forces *R* and W_1, since that part of W_2 which is transferred to *c* has no lever arm with respect to *C*. The analysis is the same as in Art. 111, and Eqs. (42) and (43) of that article are the criteria for maximum stress.

For the upper chord the conditions are different. Thus considering stress *CE*, for example, and again taking section 1, the moment center lies at *e*. That part of the load in panel *de* which is conveyed to *c* must now be included with the forces to the left, in finding the moment *M* of these forces with respect to *e*. Thus

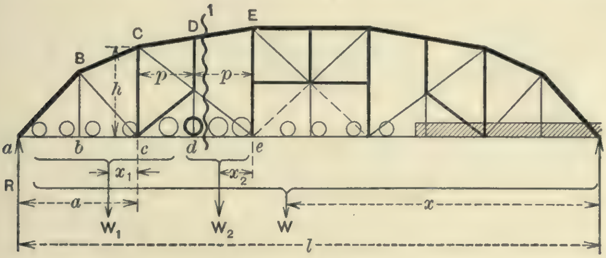


FIG. 14.

$$M=\frac{Wx}{l}a-W_1x_1-\frac{W_2x_2}{p}\times 2p. \quad . \quad . \quad . \quad . \quad (13)$$

For maximum *M*

$$\frac{dM}{dx}=\frac{Wa}{l}-W_1-2W_2=0,$$

or

$$W=\frac{l}{a}(W_1+2W_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

For the usual case of equal panels, if *n* denotes the number of panels in a, and N the total number in the truss, Eq. (14) becomes

$$W=\frac{N}{n}(W_1+2W_2). \qquad (15)$$

In seeking the position of loading which satisfies this criterion the critical weight must be placed at d. The corresponding stress in CE is then M/h', in which M may be found from Eq. (13) with the aid of the moment table.

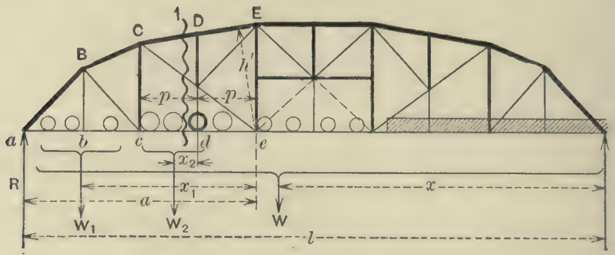


FIG. 15.

If the web system is of the type shown in Fig. 15, the conditions for the upper-chord stresses are the same as for a truss without sub-panels, and the lower-chord stresses may be analyzed as follows:

Considering stress ce, for example, and taking section 1, the moment center is seen to lie at C. The moment M about C of the forces to left of the section is

$$M=\frac{Wx}{l}a-W_1x_1+\left(\frac{W_2x_2}{p}\times p=W_2x_2\right). \qquad (16)$$

For maximum M,

$$\frac{dM}{dx}=\frac{W}{l}a-W_1+W_2=0,$$

or

$$W=\frac{l}{a}(W_1-W_2), \qquad (17)$$

and for equal panels,

$$W=\frac{N}{n}(W_1-W_2). \qquad (18)$$

In seeking the position of loading which satisfies this criterion, the critical weight must be placed at *d*. The corresponding stress in *ce* is M/h, in which M may be found from Eq. (16) with the aid of the moment table.

167. Chord Stresses in Middle Panel of a Pettit Truss with Odd Number of Double Panels. When the number of double panels is odd (Fig. 16) the analysis of the chord stresses in the middle double panel involves the consideration of shears in that panel, since the analysis depends on which of the two counters ef' and $f'g$ is in action.

Lower Chord. For the lower chord stress $ef(=fg)$ the conditions are exactly the same as for the Pratt truss without sub-panels (Art. 119). The procedure will be further illustrated in the example in the next article.

Upper Chord. For the upper chord member $EF(=FG)$ two cases must be considered. Let the critical wheel be applied at *f*, and a position be found which will satisfy the criterion $W=N/n\times(W_1+2W_2)$, Eq. (15), Art. 166, in which W_1 denotes the loading from *a* to *e* and W_2 the loading in panel *ef*. For the loading in that position compute the apex load at *f*, and assume this load to be transferred in equal parts to *E* and *G*. Compute then the live-load shear in the double panel *eg*. If this shear is positive it indicates that counter $f'g$ is in action. The conditions with respect to chord stress $EF(=FG)$ are then precisely the same as those in Fig. 14 with respect to chord stress $CD(=DE)$ on which the criterion was premised.

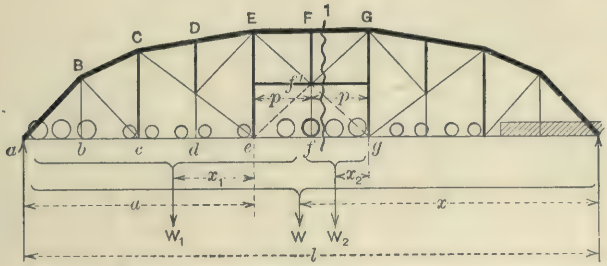


FIG. 16.

On the other hand, if for the loading headed in the same direction as before, i.e., to the left, the live-load shear in the double panel *eg* is negative, counter ef' is in action and the moment center of $EF(=FG)$ lies at e. Taking section 1, the moment about e of the forces to the left of that section is

$$M=\frac{Wx}{l}-W_1x_1+\left(\frac{W_2x_2}{p}\times p=W_2x_2\right). \quad . \quad . \quad . \quad (19)$$

This equation is seen to be the same as Eq. (16) of the preceding article, so that the criterion in this case will be the same as for Fig. 15, i.e., for equal panels, $W=N/n\times(W_1-W_2)$.

This matter will be further treated numerically in the following example.

168. Example. Live-load Stresses in a Pettit Truss. Let it be required to find the maximum live-load stresses in the trusses of the 300-ft. single-track railroad bridge, Fig. 17, from Cooper's E–60 loading.

Solution. All loads and stresses will be expressed in thousands of pounds. The values of the lever arms shown in the figure were computed in the example in Art. 163.

The analysis will be limited to the derivation of the maximum stresses in the four web members and three chord members distinguished by cross-marks. As explained in the preceding articles, the maximum stresses in the other members are found in the same manner as for a truss without sub-panels.

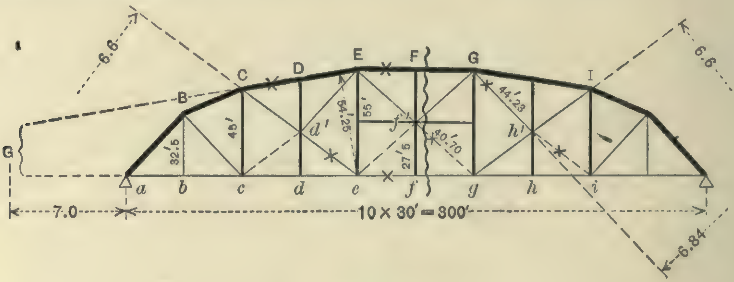


Fig. 17.

Main diagonal d'e. The criterion for maximum stress Eq. (4), Art. 165, is satisfied when $W=9/7\times5W_1(=6.4\ W_1)$. With the aid of the moment table it will be seen that this condition is met when wheel 5 is placed at e. With the loading thus placed the reaction at a is 259.5, and the forward apex load at d is 41.5, of which one-half, or 20.8, is transferred to C and the other half to E. Hence by moments about G:

$$d'e=(259.5\times7.0-20.8\times9.0)\div6.6=(+)246.9.$$

Counter h'i. The criterion for maximum stress, Eq. (4), Art. 164, is satisfied when $W=11/17\times5W_1(=3.2W_1)$. This condition is fulfilled when wheel 5 is placed at i. For that position the reaction at a is 48.3 and the forward apex load at h is 41.5, of which one-half, or 20.8, is transferred to G and the other half to I. As explained in Art. 165, the live-load stress in $h'i$ is equal in magnitude and opposite in sign to the counter stress which would be developed in gh' if $h'i$ did not exist. Thus from moments about the intersection of GI and gi,

$$h'i = (48.3 \times 17 - 20.8 \times 11) \div 6.6 = (+)89.7.$$

In Art. 162 the dead-load stress in $h'i$, when gh' is not in action, was found to be -88.6. Combining the live and dead-load stresses by the second method of Art. 102,

$$h'i = 89.7 - \tfrac{1}{2} \times 88.6 = (+)45.4.$$

Sub-diagonal Gh'. As explained in Art. 164, two conditions of loading must be considered: (1) for Gh' acting as a counter, and (2) for maximum apex load at h.

(1) *For Gh' acting as a counter*. The criterion for maximum stress, Eq. (33), Art. 136, is satisfied when $W = 11/17 \times 10W_1 (=6.5W_1)$. This condition is met when wheel 3 is placed at h. The reaction at a is then 73.4 and the forward apex load at g is 11.5. The corresponding stress may be found by moments about the intersection of GI and gi. Thus

$$Gh' = (73.4 \times 17 - 11.5 \times 11) \div 6.84 = (+)163.9.$$

The dead-load stress, $+88.6$, in gh' (Art. 163) must be neutralized before Gh' comes into play as a counter. When this occurs gh' may be regarded as removed from the truss and the dead-load stress in Gh' is then -38.5 (Art. 163). Combining the live and dead-load stresses as in Art. 102 (second method),

$$Gh' = 163.9 - \tfrac{1}{2} \times 38.5 = (+)144.6.$$

(2) *Maximum apex load at h*. The criterion for maximum floor-beam reaction, $W = 2W_1$, (Eq. (35) Art. 45) is satisfied when wheel 13 is at the floor beam. By Eq. (39), Art. 45, the reaction is

$$R = 213 - \frac{1227 + 1280}{30} = 129.4.$$

By Eq. (1), Art. 162, the corresponding live-load stress in Gh' is,

$$Gh' = 129.4 \times \frac{44.23}{55} = (+)104.1.$$

The dead-load stress in Gh' is $+47.0$ (Art. 163). Combining this with the live-load stress as in Art. 102 (second method), the resultant stress is,

$$Gh' = 104.1 + \tfrac{1}{2} \times 47.0 = +127.6.$$

This value is seen to be lower than that for condition (1).

Counter f′g. The criterion for maximum stress, Eq. (5), Art. 165, is satisfied when $W=5W_1$. This condition is fulfilled when wheel 4 is at g. The corresponding reaction at a is 127.2 and the forward apex load at f is 24.0. The latter is transferred in equal parts to E and G, without affecting the stress in $f'g$. After this assumed transfer, stress $f'g$ follows from the shear in the double panel eg. Thus

$$f'g=(127.2-12)\frac{40.70}{27.5}=(+)170.5.$$

Upper chord member CE. The criterion for maximum stress, Eq. (15), Art. 166, is satisfied when $W=\frac{10}{4}(W_1+2W_2)$. Try wheel 12 at d, bringing 170 ft. of uniform load on the span. Then

$$W=426+3\times170=936,\quad W_1=174,\quad W_2=84 \text{ or } 114.$$

Since

$$936>\frac{10}{4}(174+2\times84) \text{ and } <\frac{10}{4}(174+2\times114),$$

the criterion is satisfied. The corresponding moment about e, as expressed algebraically by Eq. (13), Art. 166, is

$$M=\frac{a}{l}Wx-W_1x_1-2W_2x_2,$$

in which the values of the moments Wx, W_1x_1 and W_2x_2 may be found with the aid of the moment table. Finally,

$$CD(=DE)=\frac{M}{54.25}=\left(\frac{4}{10}\times140{,}300-14{,}000-2\times1262\right)\div54.25=(-)729.9$$

Lower chord member ef. Placing the critical wheel at e, the criterion is $W=\frac{10}{4}W_1$ which is satisfied with wheel 15 at e. The head of the uniform load is then 9 ft. to the left of f. Let R denote the reaction at a, r the reaction (apex load) at f, and r_1 the reaction at e from the loads in panel ef. Then

$$R=434.0,\quad r=88.8 \quad\text{and}\quad r_1=41.7.$$

The sum of wheels 1 to 15, inclusive, is 367.5. The apex load, 88.8, at f is transferred in equal parts to E and G without affecting stress ef. The shear, V, in panels ef and fg is then

$$V=434-(367.5+\tfrac{1}{2}\times88.8+41.7)=-19.6.$$

Since the shear is negative, ef' is in action and the moment center of ef therefore lies at G, whereas the loading occupies a position for maximum bending moment at e.

For the critical wheel at g, the criterion is $W=\frac{10}{6}W_1$. To satisfy this, the uniform load will extend beyond g. The exact position may be found as follows:

If W_1 denotes the loading to the right of g, the criterion becomes $W=\frac{10}{4}W_1$, in which $W_1=120\times3=360$, whence $W=900$. The half-weight of the two engines is 426. The length of the uniform load is therefore $(900-426)/3=158$ ft., i.e., it must extend 38 ft. beyond g. For the loading in the position first chosen this distance was 39 ft. Since the difference is only 1 ft., it may be assumed without trial that V will remain negative. The moment center will therefore lie at G—i.e., vertically above the assumed critical point g, as it should. The moment of the entire load about the right support is 129,300, whence

$$ef=[129{,}300\times\frac{6}{10}-24{,}550-(426+38\times1.5)38]\div55=(+)630.5$$

Upper chord member EF. When ef' is in action the moment center is at e. The criterion in that case, by the preceding article, is $W=\frac{10}{4}(W_1-W_2)$. This is exactly satisfied when the uniform load extends 8 ft. beyond f, which was also found to be the governing position for ef. For that position, $W=900$, $W_1=450$ and $W_2=90$. The moment about the right support is, as before, 129,300, and wheel 18 is 17 ft. to the right of the moment center. The moment of the wheel loading about e is therefore $22{,}400-426\times17=15{,}158$. The reaction at f from the uniform load of $30\times3=90$ in panel fg is 45. Hence, from moments about e of the forces to the left of section 1:

$$EF=[129{,}300\times\frac{4}{10}-15{,}158+(8\times3\times26)+(45\times30)]\div55=(-)700.7$$

169. Tension in Posts. The determination of the exact position of a concentrated load system for maximum tensile counter stress in a given post of a Pettit truss is not warranted by practical considerations. Refinements in the assumed distribution of a load system become increasingly unwarranted as the distance covered by the loading, and therefore the number of loads, increases; and for double-track bridges as compared with single-track bridges. Again, in double-track bridges the posts are not liable to a sudden reversal of stress during the passage of a single train. As pointed out in Art. 141, the working stresses allowed in members in which the stress is not subject to reversal are usually much lower than in members subject only to stresses of the same character. For maximum unit stresses well within the elastic limit, however, this practice can hardly be regarded as rational. There is apparently no valid reason for increasing the sectional area of a compression member because it is subject occasionally to a relatively small tensile stress. For reasons stated in Art. 141, however, in designing the pin bearings at the ends of a member, account should be taken of the possibility of the sudden reversal of stress in the member by reducing the allowable working stress in bearing, i.e., the allowable unit pressure against the pin. For similar seasons, if the connections are of the riveted instead of the pin type, the allowable working stresses for rivets may be suitably reduced under the conditions just stated, which tend to cause a loosening of the rivets. This subject will receive further consideration in Vol. III.

It is evident from the foregoing that it is more important to establish the fact that the stress in a given member is subject to sudden reversal than to ascertain the exact magnitude of the counter stress.

The conditions for maximum tensile counter stress in posts *Cc* and *Ee* in Fig. 13, Art. 165, will be considered only in a general way as follows:

Post Cc. It can be shown that loads at *b* and *c* will cause tension in *Cc* irrespective of whether *ed′* or *d′c* is in action and that a load at *d* will cause compression in *Cc*. Therefore place the second or third driver of the first engine at *c* and compute the tension in *Cc*. The live and dead-load stresses in *Cc* must be found with due regard to whether *ed′* or *d′c* is in action under that loading.

Post Ee. Loads at *b* and *c* will cause tension in *Ee* provided the counter *d′c* is not brought into play. A load at *e* will produce tension and a load at *d* compression in *Ee*. By placing the second or third driver of the first engine at *e*, the probability is that the counter *d′c* will not be in action, in which case the resultant tension in *Ee* may be found by combining the live-load tensile stress for the loading in this position with the normal dead-load compressive stress in that member.

In a truss in which the sub-diagonals are ties, as in Fig. 13, instead of struts, as in Fig. 14, the loads carried to the tops of the posts by the ties cause compression in the posts, so that as a rule the resultant stress in the posts, when the loading is placed in position for maximum counter stress in these members, is compressive; or, if tensile, the resultant stress is very small.

170. Double Triangular Truss with Sub-Panels. For reasons stated in Art. 153, this truss, Fig. 18, is now seldom used. The stresses in this truss are statically determinate only if the number of double panels is even and the loads are counterbalanced. The stresses may then be analyzed as in Art. 93. The web stresses may be found by applying $\Sigma V=0$ to successive joints beginning at the middle of the truss. The chord stresses may then be computed by applying $\Sigma H=0$ to successive joints beginning at the end of the truss.

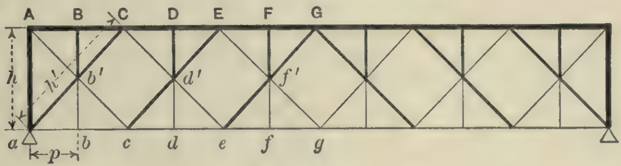


FIG. 18.

The analysis of the double triangular truss without sub-panels has been treated for dead load in Arts. 93–95, for uniform live load in Art. 109, and for concentrated live loads in Art. 122. In the truss with sub-panels the only added feature is the presence of the loads at the points of intersection of the principal web members. Considering a load W at d', for example, the proportions in which this load will be transmitted to the four apexes C, E, c and e are in general statically indeterminate. The most reasonable assumption, and one sufficiently accurate for practical purposes, is to consider $\frac{1}{4}W$ transferred to each apex. After the loads at b', d' and f' are assumed to have been thus conveyed to the principal chord joints, the analysis of the stresses may be made in the same manner as for a truss without sub-panels. The resulting stresses must then be combined with those produced by the assumed transfer of the intermediate loads from the intersection points to the chord joints. For a load W at d' these stresses are:

$$Cd'(=d'E) = +\tfrac{1}{4}Wh'/h,$$

$$cd'(=d'e) = -\tfrac{1}{4}Wh'/h,$$

$$CD(=DE) = -\tfrac{1}{4}W\times p/\tfrac{1}{2}h = -\tfrac{1}{2}Wp/h.$$

$$cd(=de) = +\tfrac{1}{4}W\times p/\tfrac{1}{2}h = +\tfrac{1}{2}Wp/h.$$

For maximum live-load stress in Cd' and $d'c$, from uniform loading by the conventional method, a full panel load wp must be applied at every apex from D to the right. For maximum stress in Ed' and $d'e$ the apexes from E to the right must be loaded.

For concentrated load systems, the web stresses may be found with sufficient accuracy by an 'equivalent' load system consisting of a series of floor-beam loads separated by intervals of one panel length, as explained in Art. 122 (*a*). For the loading advancing from the right, the first of these loads should be applied at D for maximum stress in Cd' and cd', at E for maximum stress in Ed', and at F for maximum stress in $d'e$. The chord stresses may be found by considering the above equivalent load system to extend over the entire bridge, or more simply and with sufficient accuracy by using an 'equivalent uniform load' or a modification of the same of the form described in Art. 122 (*b*).

CHAPTER XI

LIVE-LOAD STRESSES BY GRAPHIC METHODS—INFLUENCE LINES

A. Load and Moment Chart

171. Use of Load and Moment Chart. A load and moment chart for Cooper's E-60 Loading, headed by one instead of two engines to avoid confusion of lines, is shown in Fig. 1. The purpose of the *load line* is to find the position which will satisfy the criterion for maximum shear or bending moment. The *moment line* serves chiefly to determine moments, from which the reactions, shears, and bending moments may be computed in the usual way. Bending moments may also be found by direct readings, and the moment line may be used as an equilibrium polygon in determining reactions. The shear and moment lines are plotted on the same chart for convenience, although the functions of these two lines are wholly distinct and independent.

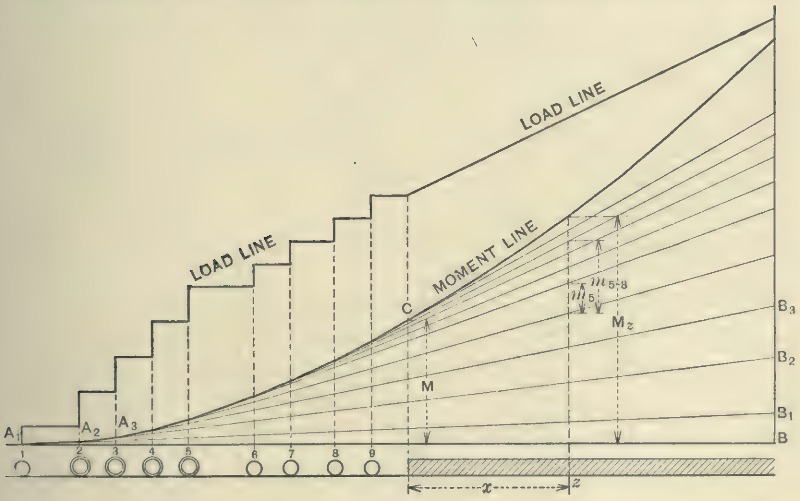


FIG. 1.

172. Construction of Load Line. Referring to Fig. 1, *the vertical ordinate to the load line from any point of the horizontal base line represents the sum of all the loads to the left of that point.* To construct the load line lay off the wheel spacing to a suitable scale (not smaller than 10 ft. to the inch) along the base line. On the ordinate over each wheel lay off the sum of all the loads to the left, including the load below the ordinate, and join these points by means of a stepped line as shown in the figure. The height of each step in the load line is equal to the load below. Above the uniform live load the slope of the load line is evidently constant, and equal to the uniform load per unit of distance. The scale for the loads should not be smaller than 50,000 lbs. to the inch.

173. Construction of Moment Line. To construct the moment line lay off on some ordinate, preferably at the extreme right end of the base line, the distances, BB_1, B_1B_2, B_2B_3, etc., representing to a suitable scale (not smaller than 5,000,000 ft.-lbs. to the inch) the moments about B of loads 1, 2, 3, etc., respectively. Then draw A_1B_1, and from its intersection with the ordinate above wheel 2, draw A_2B_2. Similarly, from the intersection of A_2B_2 with the ordinate above wheel 3, draw A_3B_3, etc. The polygonal line A_1C formed by the successive inclined lines A_1A_2, A_3A_3, etc., is the 'moment line.' Above the uniform load points on the moment line may be established by laying off vertically from any point on the base line the moment about that point of all loading to the left. This part of the moment line is a parabolic curve. For, if w denotes the intensity of the uniform load, and M the moment of the wheel loads, ΣW, about the head of the uniform load, the moment M_z at a point z, distant x from the head of the uniform load, will have the value,

$$M_z = M + \Sigma Wx + \tfrac{1}{2}wx^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

which is the equation of a parabola.

It is thus seen that in general *the vertical ordinate to the moment line from any point on the base line represents the moment about that point of all loads to its left.*

Again, from the construction of the diagram, it is clear that the vertical intercept m_5 represents the moment of wheel 5 about z, and the intercept m_{5-8}, the moment of wheels 5 to 8, inclusive, about z.

174. The Moment Line as an Equilibrium Polygon. The moment line described in the preceding article is an equilibrium polygon, and the method of obtaining the moments is in conformity with the principle derived in Art. 37 (Third Method) as may be seen from the following:

In Fig. 2(*a*) the moment line A_1C is constructed as explained in the preceding article. At a unit distance from A_1 the ordinate from the base line to line 1 equals w_1. At a unit distance horizontally from A_2 the ordinate from a horizontal line through A_2 to line 2 equals $w_1 + w_2$. That is, the slopes of lines 1 and 2 are w_1 and $w_1 + w_2$, respectively. Similarly, the slope of line 3 is $w_1 + w_2 + w_3$, etc. Therefore, if the loads w_1–w_5 are laid off as in Fig. (*b*), and if the lines 0–5 in that figure are drawn parallel to the corresponding lines in Fig. (*a*), the former must intersect at a common point O. This point is the pole, and the lines 0–5 radiating from the pole are rays to the force polygon *a*–*f*. The distance Oa is the pole distance. The forces F and F_1 are imaginary forces which hold the loads w_1–w_5 in equilibrium.

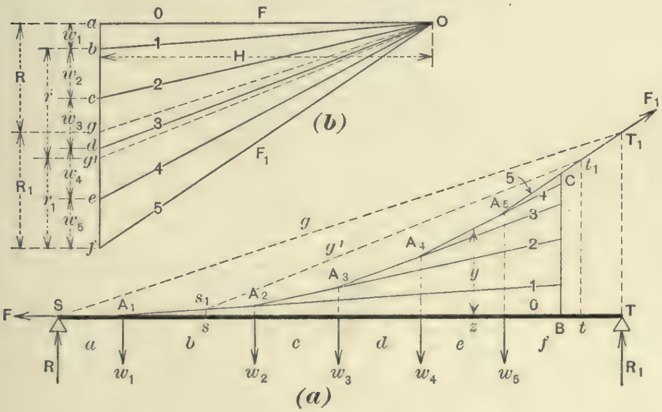


Fig. 2.

Again, in Fig (*a*) any segment, as A_4A_5, for example, represents the line of action of the resultant of all the forces to the left—in this case, forces F and w_1–w_4. By the principle derived in Art. 37, (Third Method) the moment M_z of these forces about any point, as z, for example, may be found by multiplying the vertical distance y, from that point to the resultant, A_4A_5, of the forces, by H, the pole distance Oa, i.e.,

$$M_z = Hy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since z is a point on the line of action of F, the moment of that force about z is zero. Therefore, $M_z(=Hy)$ is equal to the moment of loads w_1–w_4 about z.

In applying Eq. (2), y must be measured by the same scale to which the distances between successive wheels are laid off in Fig. (*a*), and H (which is a *force,* although called the pole *distance*) by the scale used for the loads in Fig. (*b*). Suppose that Fig. (*b*) is drawn first, and Fig. (*a*) is based on Fig. (*b*); then, if H is laid off as 100,000 lbs., and the wheel spacing to a scale of 1 in. = 10 ft., $M_z = y$ (in inches) $\times 10 \times 100{,}000 = 1{,}000{,}000\ y$ ft.-lbs. The ordinates, y, may thus be regarded as representing the moments to a scale of 1 in. = 1,000,000 ft.-lbs. Suppose, however, that Fig. (*a*) is drawn first, in the manner explained in Art. 173, and that the vertical ordinates represent the moments themselves to a scale of 1 in. = 5,000,000 ft.-lbs. Then, if the wheel spacing is laid off as before to a scale of 1 in. = 10 ft., the value of H in Fig. (*b*), constructed as above described, will be five times 100,000 lbs. (= 500,000 lbs.) when measured by the scale to which the forces in that figure are laid off. That is to say, if in Eq. (2) y is decreased fivefold, the value of H must increase in like ratio in order that M_z may have the same value, as it should. Thus it is seen also that if Fig. (*a*) is drawn first, the value of H is predetermined by the relation between the scales used for wheel spacing and for moments.

In general, if the distance scale is 1 in. = n ft., and the moment scale, 1 in = m ft.-lbs., $Hn = m$, or

$$H \text{ (in lbs.)} = \frac{m}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the moment line is laid off as explained in the preceding article, there is obviously no need of drawing a figure similar to Fig. 2 (*b*), nor to determine H by Eq. (3). For in that case the ordinates represent foot-pounds by the moment scale $m(=Hn)$, and not feet by the distance scale n. Instead of determining the moments as the products of the y's and H, these products are scaled off directly.

In graphic methods a clear understanding of the interrelations of the scales, actual or implied, is of fundamental importance.

175. Reactions by Graphic Method. Shear in a Beam. Referring to Fig. 2 of the preceding article, the reactions from that loading, or any part of that loading, may be found graphically by means of the moment line, Fig. (*a*), and an auxiliary diagram like that shown in Fig. (*b*).

Let it be required, for example, to find the reactions R and R_1 at the supports S and T, Fig. (*a*), of a simple beam or truss, from the loads w_1–w_5 when these loads occupy the position relative to the supports shown in the figure. From T draw the vertical TT_1 to its intersection with segment 5. Draw ST_1, the closing line of the equilibrium polygon, and in Fig. (*b*) draw Og parallel to ST_1. The desired reactions R and R_1 are then ga and fg, respectively. Similarly, if the supports are at s and t, the reactions r and r_1 at these supports from the loads w_2–w_5 may be found by drawing Og' in Fig. (*b*) parallel to the closing line s_1t_1 in Fig. (*a*).

After the reactions have been found the shear at any section of a beam may be determined in the usual manner by subtracting from the reaction at either support the sum of the loads between that support and the section in question.

176. Maximum Live-load Shears in a Truss. Consider, for example, a truss of six equal panels. Lay off the panel lengths, *ab*, *bc*, etc., on a strip of paper, Fig. 3 (*b*), to the same scale as the horizontal scale of the load and moment chart, Fig. (*a*). Let it be required to find the maximum live-load shear in panel *bc*. First, find the position of the loading which satisfies the criterion $W_1/p = W/l$, Eq. (39), Art. 111, in which W_1 denotes the load in the panel and W the total load on the bridge. Try wheel 3 by placing it vertically above c, as in the figure. Then either CC' or CC'' represents the load in panel *bc*, according to whether wheel 3 is applied a differential distance to the right or left of c; and BB' represents the load on the bridge. The ratio W/l is represented by the slope of line AB', and the ratio W_1/p by the slope of line DC' or DC'', according to whether wheel 3 is assumed to act immediately to the right or left of c. If DE, drawn from D parallel to AB' intersects $C'C''$, the criterion is satisfied, since the value of the ratio W/l is then intermediate between the two values of the ratio W_1/p. In applying this method, it is evidently not necessary to draw the lines DC' and DC''. After locating the points A, B', and D, apply a triangle so that one of its edges traverses AB', and then slide the triangle till this edge, its direction remaining parallel to AD', traverses D.

Finally, the shear, V, in panel bc may be found from the ordinates to the moment line at B and C, i.e.,

$$V = M/l - m/p. \qquad (4)$$

177. Maximum Live-load Web Stresses in a Truss with Inclined Chords. The load line may be used as an aid in finding the position of a concentrated load system for maximum stress in the web members of a truss with inclined chords by a method

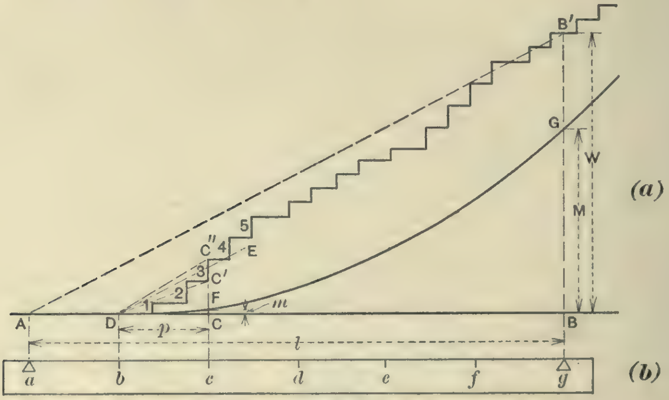


FIG. 3.

to be described later (Art. 194). After the position has been determined, the reaction, M/l, at the left support of the truss, and the forward joint load, m/p, may be computed from the moments M and m obtained from the moment line, and finally the stress itself may be found in the usual manner (Art. 136).

178. Maximum Live-load Bending Moments. Let it be required to find the maximum live-load bending moment in a beam or truss at any section distant a, Fig. 4, from the left support. Lay off the span l and the distance a on a strip of paper, Fig. (b), to the same scale as the horizontal scale of the load and moment chart, Fig. (a). Then find the position of the loading which satisfies the criterion $W_1/a = W/l$ in which W_1 is the load for the distance a and W the total load on the bridge. This criterion applies (1) to beams (Eq. (28), Art. 42); (2) to the bending moments at joints of the loaded chord of trusses in general (Eq. (42), Art. 111); and (3) to the bending moments at joints of the unloaded chord of trusses with vertical web members. Try wheel 6 above c. Then W/l is represented by the slope of line AB', and W_1/a by the slope of line AC' or AC'', according to whether wheel 6 acts immediately to the right or left of c. Therefore, the criterion is satisfied if AB' intersects $C'C''$. The lines AC' and AC'' need not be drawn. After locating the points A and B' the fulfillment of the criterion may be conveniently ascertained by stretching a silk thread between these points.

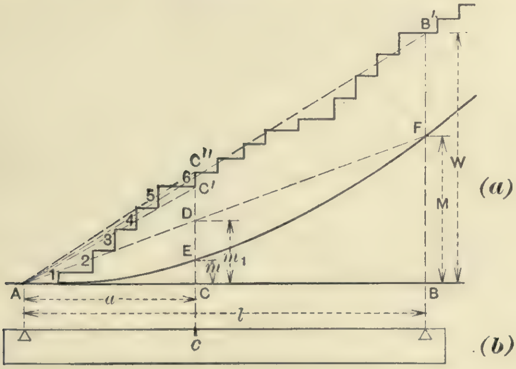


FIG. 4

The bending moment, M_c, at c is then $Ma/l - m$, in which the moments M and m are represented by the ordinates to the moment line at B and C respectively. But Ma/l, the moment about c of the reaction at A, is equal to the ordinate CD. Letting m_1 denote this moment,

$$M_c = m_1 - m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which the values m_1 and m may be read directly from the chart.

In the general case, shown in Fig. 5, some of the wheels on the left may be off the span when the loading occupies the position for maximum bending moment at c. By analogy with the foregoing the criterion is satisfied when $A'B'$ intersects $C'C''$, and the bending moment at c may be found, as before, by Eq. (5), in which M_c is now the difference of the readings m_1 and m in Fig. 5. Since the values of the ordinates to the moment line are given in such charts with reference to the horizontal base line, the value of M_c in the present case may be found more conveniently as the difference between m_1' and m'.

The above method is not adapted to the bending moments at the joints of the unloaded chord of a truss without vertical web members (Art. 121), nor to the determination of the position for maximum bending moment at such a joint. Since such trusses are, however, seldom or never used for railroad bridges, this case will not be treated.

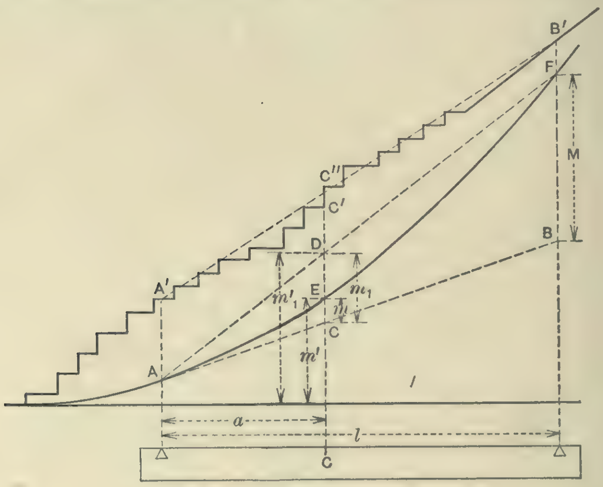


FIG. 5.

179. Another Form of Moment Line is shown in Fig. 6. This line is constructed as an equilibrium polygon. The loads are first laid off to scale along the line PQ, beginning at the bottom. The pole O is then selected, using for the pole distance, H, a quantity based on a convenient scale for moments. Thus, if it is desired to use the distance scale 1 in. $=n$ ft., and the moment scale 1 in. $=m$ ft.-lbs., $H=m/n$ (Eq. (3), Art. 174). The equilibrium polygon is then drawn, placing wheel 1 on the vertical through O, and letting the first segment coincide with ray OP. The equilibrium polygon for the uniform load will be a parabola. It may be constructed by dividing the uniform load into sections of 5 or 10 ft., and considering the load in each section to be concentrated at the center of that section. The equilibrium polygon will then be a series of tangents to the true equilibrium polygon, which can readily be drawn.

Only one equilibrium polygon is shown in Fig. 6, but in practice it is convenient to draw two, or perhaps three polygons, to different scales, and including different portions of the load system. Thus, one diagram might be drawn for the two engines, and another for the two engines and 100 or 150 ft. of uniform load.

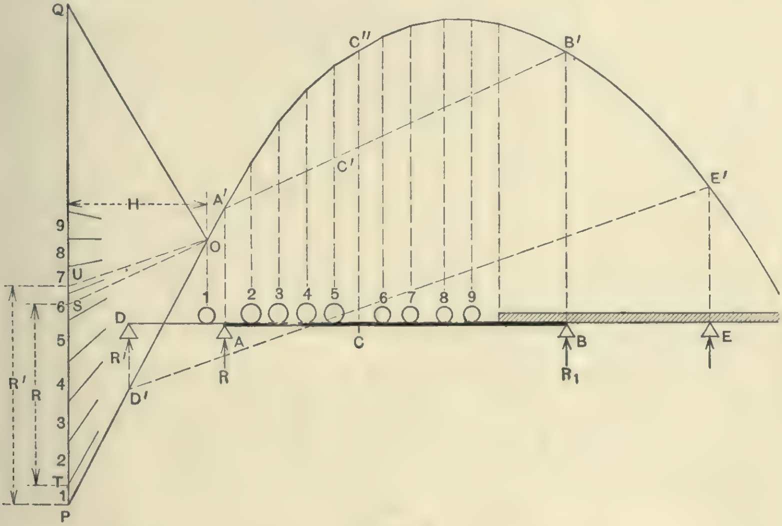


FIG. 6

Reactions. Suppose it is required to find the reactions for a beam of span AB when the loads are placed as shown. Draw verticals through A and B until they intersect the equilibrium polygon at A' and B' respectively. Then the line joining A' and B' is the closing line of the equilibrium polygon. Drawing OS parallel to $A'B'$, $R = TS$. When the uniform load is partly on the span, R_1 can best be determined by subtracting R from the total load on the beam.

Similarly, for a span DE the closing line is $D'E'$, and OU, drawn parallel to $D'E'$, determines the reaction $R'(=PU)$ at the left end.

Bending Moments. If it is required to find the bending moment, M_C, at some point C of the beam AB, draw a vertical through C until it intersects $A'B'$ at C' and the equilibrium polygon at C''. Then $M_C = C'C'' \times H$ (Art. 37*b*), in which $C'C''$ is measured by the scale of distances. If H has been properly chosen, as explained above, M_C may be found by measuring $C'C''$ directly by the moment scale, i.e., $M_C = C'C''$.

B. Influence Lines.

180. Definition of Influence Line. Any effect produced by a single concentrated load (assumed for convenience at *unity*) on a beam or truss, which can be expressed quantitatively as a function of the *position* of the load, may be represented for all positions of the load by means of an *influence line.* The influence line is so drawn with reference to a horizontal base line on which the span is laid off, that the ordinate to the influence line at any point of the span represents the value of the function when the unit load is applied at that point. The value of the function for any other load, W, applied at any point of the span may then evidently be found by multiplying the ordinate at that point by W. Similarly, by the summation of such products the value of the function for any series of loads may be determined.

181. Influence Line for Reaction. Let it be required to draw the influence line for the left reaction, R, of a simple beam. The general equation for R, Fig. 7, for a load of unity in any position is

$$R = \frac{x}{l}. \qquad (6)$$

This equation is that of a straight line. For $x=0$, $R=0$, and for $x=l$, $R=1$. Laying off $AA'=1$, $A'B$ is the desired influence line for the left reaction. The ordinate, x/l, at any section distant x from the right support is equal to the reaction from a unit load at the section in question.

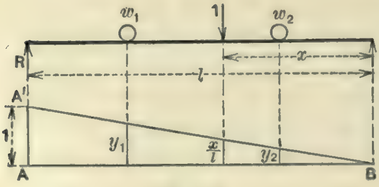


Fig. 7

(*a*) *Concentrated Loads.* The reaction from a concentrated load in any position is evidently equal to the product of the load by the corresponding ordinate to the influence line. Thus the reaction R for the two loads, w_1 and w_2, in the positions shown is

$$R = w_1 y_1 + w_2 y_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

(*b*) *Uniform Load.* A uniform load of intensity w may be regarded as consisting of a series of equal concentrated loads $w \cdot dx$ separated by differential distances dx. In Fig. 8, let y denote the ordinate to the influence line at one of these differential loads $w \cdot dx$. The corresponding differential reaction, dR, is then $w \cdot dx \times y$, in which $dx \times y$ equals the differential area dA in the figure. Thus $dR = w \cdot dA$, and since this is true of every differential load $w \cdot dx$, the total reaction is

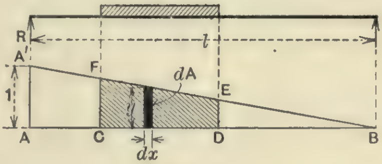


FIG. 8

$$R = Aw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

in which A denotes the area of the shaded trapezoid $CDEF$.

But A is equal to $\overline{CD}$ multiplied by the ordinate at the middle of CD. Hence, in general, *the value of the function from a uniform load extending over any distance for which the influence line is a straight line, is the product of the total uniform load for that distance by the ordinate to the influence line at the middle of that distance.* When the uniform load covers the entire span, the area A, and therefore R, becomes maximum. In that case A denotes the area, $\frac{1}{2} \times l$, of the triangle ABA', whence $R_{\max} = \frac{1}{2} wl$.

182. Influence Line for Shear in a Beam. Consider the shear, V, at section z of a beam, Fig. 9, from a unit load as it moves across the span from right to left, i.e., for values of x ranging from zero to l.

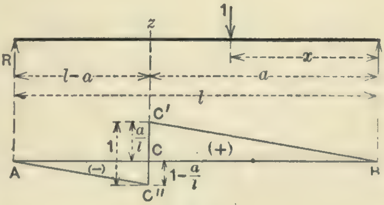


FIG. 9

For values of x from zero to a, $V(=R)$, is positive, or in general, for $x < a$,

$$V = +\frac{x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

which is the equation of a straight line. When the unit load is just to the right of z, $x=a$ and $V=+a/l$. For $x=0$, $V=0$. The influence line, $C'B$, for the segment of the beam to the right of z may therefore be drawn by laying off $CC'=a/l$, and connecting C' and B by a straight line.

For values of x from a to l, $V(=R-1)$ is negative, i.e., $V=-(1-R)$, or in general, for $x>a$,

$$V=-\left(1-\frac{x}{l}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When the unit load is just to the left of z, $x=a$, and $V=-(1-a/l)$. Thus, as the load passes z from right to left the shear at that section is suddenly reduced from a/l to $a/l-1$ or $-(1-a/l)$, i.e., it is reduced by an amount equal to the magnitude, unity, of the load. For $x=l$, $V=0$. Hence, to complete the influence line, lay off $C'C''=1$, and draw AC''. As a check, observe that AC'' and $C'B$ are parallel, since these lines have the common slope $1/l$. A convenient construction is to lay off an ordinate unity (not shown) above A, and to draw $C'B$ (slope, $1/l$) as part of the line from B to the extremity of that ordinate and AC'' parallel to $C'B$.

(*a*) *Concentrated Loads.* Let it be required to find the maximum shear at section z of a beam, Fig. 10, from Cooper's E-60 loading.

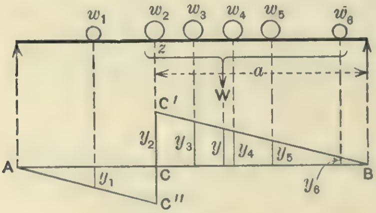


FIG. 10.

Construct the influence line $AC''C'B$ as just described. It has been shown (Art. 40*c*) that for $a>\frac{1}{2}l$, the shear at section z is maximum when, with the engine headed toward the left, the first driver (second wheel) is placed at (immediately to the right of) the section, as shown in the figure. Then

$$V_z\,(\max)=(w_2y_2+w_3y_3+\ldots+w_6y_6)-w_1y_1. \quad . \quad . \quad (11)$$

In general the summation of the products of individual concentrated loads and the corresponding ordinates to the influence line is equal to the product of the resultant of the loads and the corresponding ordinate at the center of gravity of the loads, provided the influence line is *straight* for the distance over which the loads in question extend.

Thus, if $C'D'$, Fig. 11, is some straight influence line for the panel CD of a beam or truss, $w_1y_1+w_2y_2+w_3y_3+w_4y_4=Wy$ if W is the resultant of the loads, in magnitude and position. For if x_1, x_2, x_3, and x_4 denote the horizontal distances of w_1, w_2, w_3, and w_4, respectively, from G, the intersection of $C'D'$, produced, with the base line, $w_1x_1+w_2x_2+w_3x_3+w_4x_4=Wx$. And, since $y_1/x_1=y_2/x_2=\ldots\ldots=y/x$, it follows that

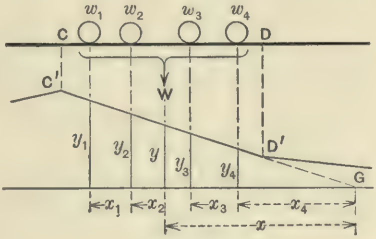


Fig. 11

$$w_1y_1+w_2y_2+\ldots\ldots+w_ny_n=Wy.$$

Eq. (11) may therefore be expressed as follows:

$$V_z(\max)=Wy-w_1y_1. \quad \ldots\ldots \quad (12)$$

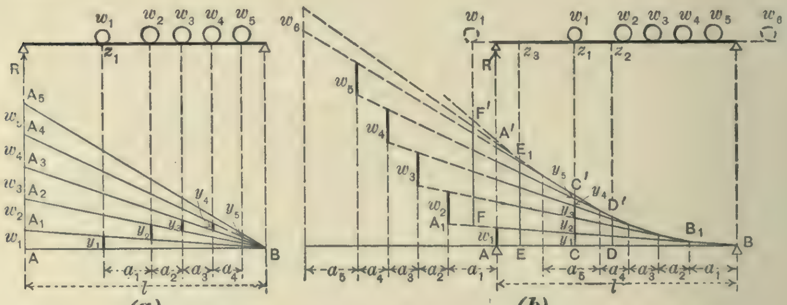
In general the position of the load system for V_z(max) must be be found by trial. It is evident, however, by inspection of the influence line in Fig. 10, that for maximum shear at z, (1) the loads must be distributed over the larger segment of the span; (2) the heaviest loads must be placed within the region where the ordinates to the influence line are longest; (3) one of the heaviest loads must be placed immediately to the right of the section; and (4) the loading on the smaller segment must be relatively small or zero. In the usual engine loading the first load, w_1, is relatively small, and its distance from the next load is relatively large. For such a load system advancing from the right, the positive shears in the left half of the beam are almost invariably maximum when the second wheel (first driver) is placed at (immediately to the right of) the section (Art. 40*c*).

(*b*) *Uniform Load.* For a uniform load, the shear at section z may be found, as explained in Art. 181 (*b*), as the product of w, the intensity of the uniform load, and A, the area between the base line and the influence line bounded by the two ordinates at the extremities of the uniform load. For maximum positive shear at section z it is seen from the influence line, Fig. 9, that the uniform load must extend over the distance a. The corresponding shear at z is equal to the area of the triangle BCC', multiplied by w, i.e., $V_z(+\max) = \frac{1}{2}a^2w/l$, which agrees with Eq. (14), Art. 40.

183. Reactions by Reaction Polygon. (*a*) *Concentrated Loads.* On the base line AB, Fig. 12 (*a*), erect a vertical at A, and lay off the loads w_1–w_5 as shown. Then if the vertical intercepts, y_1–y_5, at the loads are measured by the same scale to which the loads were laid off, the reaction at the left support from the loading in the position shown is evidently equal to the sum of these intercepts, i.e.,

$$R = y_1 + y_2 + \ldots + y_5. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From Art. 181, A_1B is the influence line of w_1, drawn for w_1 direct, instead of for a unit load. In a similar sense A_2B is the influence line of w_2 with respect to A_1B as a base line; A_3B is the influence line of w_3 with respect to A_2B as a base line; etc.



(*a*) (*b*)

FIG. 12

It will now be shown that by constructing the *reaction polygon* $A'B$, Fig. 12 (*b*), the reaction R, when w_1 is at z_1, is represented by the ordinate CC' to the reaction polygon, at w_1; and further that for any other position, so long as w_1 remains on the span, the ordinate at w_1 will represent the reaction R.

In Fig. (*a*), in order that y_2 may be brought vertically above y_1, the triangle BA_1A_2 must be slid to the left a horizontal distance, a_1, equal to the distance of load w_2 from w_1. Similarly, to bring y_3 vertically above y_1 the triangle BA_2A_3 must be slid to the left a horizontal distance, $a_1 + a_2$, equal to the distance of load w_3 from w_1. The results of such movements are shown in Fig. (*b*), the construction of which is self-explanatory. Observe that the load spacing, a_1, a_2, etc., in Fig. (*b*) is laid off from B to the left, and the direction in which the spaces follow each other is therefore opposite to that in Fig. (*a*). Evidently CC' represents the reaction R for the loading in the position shown, with w_1 at z_1.

If the load system is moved to the right until w_1 is at z_2, the corresponding ordinate, DD', to the reaction polygon will represent the reaction R for the loading in that position, w_5 being now off the span. If the load system is moved to the left until w_1 is at z_3, thereby bringing w_6 on the span, the ordinate EE_1 to the reaction polygon will represent the reaction R for the loading in that position. If the load is advanced to the left until w_2 is at z_3, w_1 will go off the span to the position indicated, Let it be assumed that by this movement another load, w_7, will come on at the right. The reaction R is now represented by the ordinate FF' to the reaction polygon, measured from the new base line A_1B_1, at w_1. This intercept is the same as that at w_2 (now the leading load) in its new position at z_3, if the span is laid off with the right support vertically below B_1, this intercept being also measured from A_1B_1.

(*b*) *Uniform Load.* If a uniform load of intensity w extends a distance x from the right support, the reaction at the left support has the value $R=\frac{1}{2}wx^2$, which is the equation of a parabola, convex to the base line AB, Fig. 12 (*b*), with vertex at B.

184. Shear in a Beam by Reaction Polygon. The reaction at the left support of a beam for the loading in any position may be found by means of a reaction polygon, as explained in the preceding article. The shear at any section may then be obtained in the usual way by subtracting from the left reaction the sum of the loads between the left support and the section in question.

185. Reaction Polygon as a Moment Line. Comparing Fig. 12 with Fig. 2, Art. 174, it is seen that the construction of the reaction polygon is similar to that of the moment line, the differences being as follows:

1. In laying off the reaction polygon the pole distance, AB, Fig. 12 (*a*), is given the same length as the span, Fig. (*b*), whereas for the moment line, the pole distance, Eq. (3), Art. 174, is independent of the span.

2. The loads are laid off in successive order upward, Fig. 12 (*a*), for the reaction polygon, and downward, Fig. 2 (*b*), for the moment line. Consequently, in general appearance the reaction polygon, Fig. 12 (*b*), is the same as the moment line, Fig. 2 (*a*), in reversed position.

The reaction polygon is thus seen to be a moment line of a special kind adapted only to the particular span for which it is drawn. In the general moment line, Fig. 2 (*a*), the slopes of the successive inclined lines 1, 2, 3, etc., are w_1, w_1+w_2, $w_1+w_2+w_3$, etc., respectively (Art. 174), whereas in the reaction polygon the slopes of the corresponding lines are w_1/l, $(w_1+w_2)/l$, $(w_1+w_2+w_3)/l$, etc., respectively. Therefore, if the moment scale for the general moment line is the same as the load scale for the reaction polygon, the ordinates at corresponding points of the base lines will be l times as long for the former as for the latter. Conversely, the pole distance for the latter will be l times as long as for the former. The resulting moments, Hy, Eq. (2), Art. 174, will therefore have the same values at corresponding points, in both cases, as they should.

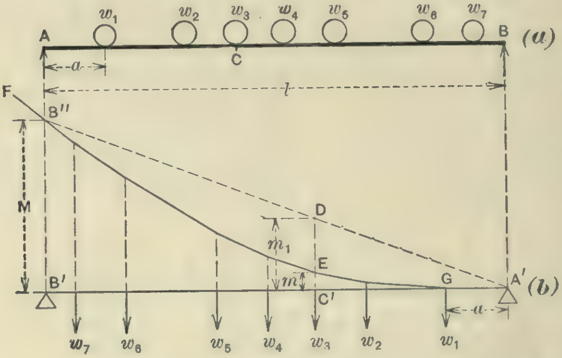


FIG. 13

186. Bending Moments by Reaction Polygon. Let it be required to find the bending moment at C, Fig. 13 (*a*), for the loading in the position shown with w_3 at C. Let FG represent the reaction polygon drawn for the span l, as explained in Art. 183. In using the polygon as a moment line, the loads must be considered in their true position with reference to the same, as shown in Fig. (*b*). Since the loading is then reversed, consider the span to be also reversed, so that the distance a is now measured from the right support, A'. Apply the reaction polygon to $A'B'$ with w_1 at the same distance, a, from A', as the distance from that load to A in Fig. (a). The bending moment at C', Fig. (b), is then equal to the required bending moment at C, Fig. (a).

In the preceding article the reaction polygon was shown to be a moment line, and therefore an equilibrium polygon, of a particular kind. The bending moments may therefore be found in precisely the same manner as explained for the moment line in Art. 178. The analogy between Fig. 4 of that article and Fig. 13 is complete, except that in the latter the ordinates M, m_1, and m are expressed in pounds instead of foot-pounds, and these ordinates must be multiplied by the pole distance, which in this case is equal to the span-length l expressed in feet. Observe that in the general expression, Hy, for bending moments, the ordinates y are now measured by a force scale instead of a distance scale, and the pole distance H is measured by a distance scale instead of a force scale, the values of the products Hy thus remaining unaffected. The bending moment, M_c, at C, in foot-pounds, is then by analogy with Eq. (5), Art. 178,

$$M_c = (m_1 - m)l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

187. Comparison of Graphic Methods. For finding reactions and shears the reaction polygon is more convenient than the general moment line, in that by the former the reaction may be scaled directly, whereas by the latter the moment, M, of the total load about the right support must be divided by the span length. For bending moments, however, the moment line is more convenient, since it avoids the multiplication of the scaled values by l (Eq. 14).

Finally, it is important to observe that a general moment line is applicable to a span of *any* length whereas a reaction polygon is applicable only to the particular span length for which it is drawn.

188. Influence Line for Bending Moment in a Beam. Consider the bending moment, M, at a section z of a beam, Fig. 14, from a unit load as it moves across the span from right to left, i.e., for values of x ranging from 0 to l.

For $x < a_2$, $M = Ra_1$, or

$$M = \frac{a_1 x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is the equation of a straight line.

When the unit load is at z, $x = a_2$ and $M = a_1 a_2 / l$. For $x = 0$, $M = 0$. The influence line $C'B$ for the segment of the beam to the right of z may therefore be drawn by laying off $CC' = a_1 a_2 / l$, and connecting C' and B by a straight line. The slope of this line is a_1 / l, so that it may be conveniently drawn by laying off a_1 vertically above A and connecting the extremity of this ordinate (not shown) with B.

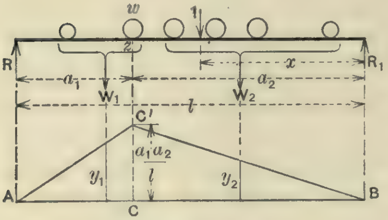


Fig. . 14

For $x > a_2$, $\qquad M = R_1 a_2,$

or

$$M = \frac{a_2(l - x)}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

By this equation, for $x = a_2$, $M = a_1 a_2 / l$, as before, and for $x = l$, $M = 0$. Hence the influence line may be completed by drawing the straight line AC'.

(*a*) *Concentrated Loads.* Suppose that a concentrated load system is applied to the beam with some load, w, at z. Then if W_1 and W_2 denote the resultants, in magnitude and in position, of the loads to either side of z, the bending moment, M_z, at z has the value

$$M_z = W_1 y_1 + W_2 y_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The load w at z may evidently be included with W_1, as in the figure, or with W_2, with due regard to the corresponding changes in y_1 and y_2. For maximum bending moment at z, the criterion (Eq. (28), Art. 42), must be satisfied, i.e., $W_1 + W_2 = W_1 \times l / a_1$.

(*b*) *Uniform Load.* For a uniform load, the bending moment at section z may be found, as explained in Art. 181 (*b*), as the product of w, the intensity of the uniform load, and A, the area between the base line and the influence line bounded by the two ordinates at the extremity of the uniform load. For maximum bending moment it is seen from the influence line, Fig.

14, that the uniform load must extend over the entire span. The corresponding bending moment at z is equal to the area of the triangle $AC'B$, multiplied by w, i.e., M_z (max) $=\frac{1}{2}a_1a_2w$, which agrees with Eq. (21), Art. 42.

189. Use of Influence Lines. It has been seen that influence lines may be used (1) to show graphically the variation of a given function with the position of a single concentrated load, and (2) to obtain the value of that function for any loading. It will now be shown that influence lines may also be used (3) for deriving the criterion for determining the position of the loading for which the value of the function in question will be maximum.

190. Criterion for Maximum Bending Moment at Any Section of a Beam, by Influence Line. The criterion for maximum bending moment at any section, z, of a simple beam, Fig. 15, may be derived by means of the influence line for bending moment at that section, as follows:

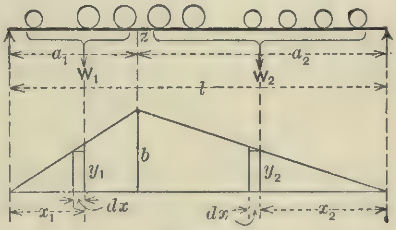


FIG. 15

Let M denote the bending moment at z. Then for the loading in the position shown,

$$M=W_1y_1+W_2y_2,$$

in which $y_1=bx_1/a_1$ and $y_2=bx_2/a_2$, whence

$$M=\frac{W_1b}{a_1}x_1+\frac{W_2b}{a_2}x_2.$$

If the loading advances a differential distance, dx, toward the left, without causing any load to cross section z or either end support,* x_1 is decreased and x_2 is increased by dx. The resulting differential change, dM, in M is

$$dM=-\frac{W_1b}{a_1}dx+\frac{W_2b}{a_2}dx.. \quad . \quad . \quad . \quad . \quad (18)$$

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* For a more critical treatment of the mathematical significance of the criterion for maximum, see Arts. 112 and 197.

If dM is plus, M will be *increased* by a movement to the *left,* whereas if dM is minus, M will be decreased by such a movement, and therefore *increased* by a movement to the *right.* Hence, for $M_{\max}$, $dM=0$. Further, aspreviously shown (Art. 42*c*) for $M_{\max}$ one of the loads must be applied at the section in question. In Eq. (18), letting $dM=0$ and reducing,

$$\frac{W_1}{a_1}=\frac{W_2}{a_2}. \qquad (19)$$

If $a_2=na_1$, then $W_2=nW_1$, whence

$$\frac{W_1}{a_1}=\frac{nW_1}{na_1}=\frac{(n+1)W_1}{(n+1)a_1}=\frac{W_1+W_2}{a_1+a_2},$$

or, letting $W=W_1+W_2$, and $l=a_1+a_2$,

$$W=\frac{l}{a_1}W_1. \qquad (20)$$

This is the desired criterion for maximum bending moment at any section of a beam, and it is seen to agree with Eq. (28), Art. 42.

Referring to Eqs. (19) and (20), it is seen that these criteria for determining the position of loading which will cause $W_1y_1+W_2y_2$, and therefore M—the function represented by the influence line—to become maximum, are entirely independent of the value of the ordinate b to the influence line at z. Hence *these criteria are applicable to every case in which the influence line consists of two inclined segments as in Fig. 15.*

191. Influence Line between Successive Loaded Panel Points. The live loading can usually be communicated to the trusses only at the joints of the loaded chords, to which it is transmitted by the floor beams. Similarly, in the case of plate-girder bridges, the floor system is frequently connected with the main girders only at intervals of a panel length, i.e., at the panel points. It will be shown now that under such circumstances the influence line is a *straight line* between successive panel points.

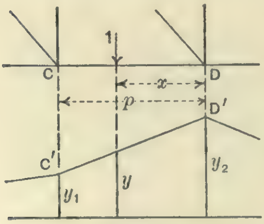

FIG. 16

Thus, on the influence line partly shown in Fig. 16, let C' and D' be points corresponding to the loaded panel points C

and D, respectively. If the unit load occupies the intermediate position shown, the loads transmitted by the stringers to C and D are respectively x/p and $(p-x)/p$. Therefore, the ordinate, y, to the influence line, corresponding to the unit load in the position shown, is

$$y=\frac{x}{p}y_1+\frac{p-x}{p}y_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

In the equation y and x are the only variables, and it is therefore the equation of a straight line. For $x=0$, $y=y_2$; and for $x=p$, $y=y_1$. Hence, for values of x between the limits zero and p the values of y by Eq. (21) will be represented by the ordinates to the straight line $C'D'$. Thus it is seen that the influence line may be completely drawn by determining the ordinates to this line at successive joints, and then connecting the extremities of these ordinates by a series of straight lines.

192. Influence Lines for Shear in a Truss. Consider the shear V in panel cd of the truss shown in Fig. 17 (a) from a single concentrated load as it moves across the bridge. If the load is to the right of d, $V=+R$; if the load is to the left of c, $V=-R_1$; if the load is in panel cd, $V=R-r$, r denoting the forward joint load at c. For the load in any position outside the panel cd, the value of V is precisely the same as the shear for the corresponding distance cd in a simple beam of span l. The shear V in panel cd of the truss when the load is in one of the end panels is not affected by the circumstance that the load is then transmitted in part directly to the abutment by the stringers in that panel. Thus suppose that a load W is in end panel gh at a distance z from the right support. If r denotes the forward joint load at g, $r=Wz/p$, and $R(=V)=rp/l=Wz/l$, which is seen to be equal to the left reaction of a simple beam for a load W applied at a distance z from the right end.

Thus it is evident that the segments AC' and $D''H$, Fig. (b), of the influence line $AC'D''H$ for V, are precisely the same as for a simple beam. These lines may be drawn as explained in Art. 182, by laying off the ordinate $AA''=1$, connecting A'' and H, and drawing AC' parallel to $A''H$. Then by Art. 191, the influence line may be completed by the straight line $C'D''$.

(a) *Uniform Load.* From the influence line it is seen that for maximum positive shear V, the uniform load must extend over the distance HQ, i.e., a distance x into the panel in question. Let it be required to find the value of x by means of the influence line.

The notation in Fig. (a) is the same as in Fig. 44, Art. 124. Evidently $\overline{CC'} = m/N$, and $\overline{DD''} = n/N$. Hence

$$x = \frac{n/N}{(n+m)/N} p,$$

and, since $n+m = N-1$,

$$x = \frac{n}{N-1} p, \qquad (22)$$

which agrees with the equation following Eq. (57), Art. 124.

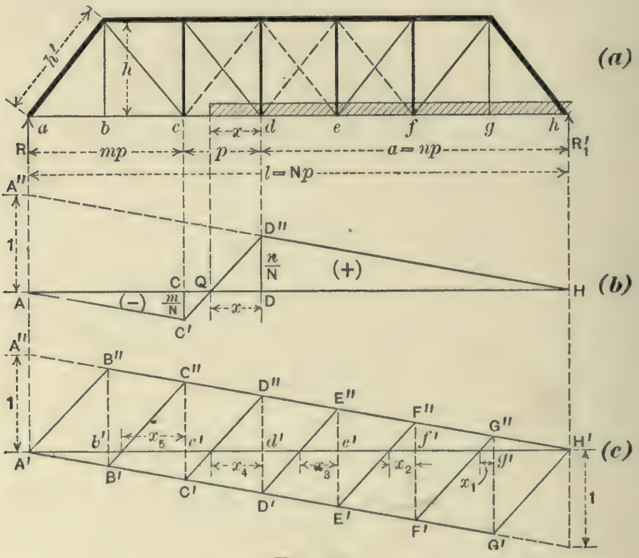

FIG. 17

The value of the maximum positive shear in panel cd is found by multiplying the area of the triangle $QD''H$ by the intensity, w, of the uniform load, i.e.,

$$V_{\max} = \left(n + \frac{n}{N-1}\right) p \times \frac{n}{2N} \times w,$$

or

$$V_{\max} = \frac{wp}{2(N-1)} n^2, \qquad (23)$$

which is seen to agree with Eq. (57), Art. 124.

The influence lines for the shears in the individual panels may be combined in a single figure, as shown in Fig. (*c*), whose construction is now self-evident. Thus the influence line for the shear in panel *ef* is $A'E'F''H'$. The influence lines for the shears in the end panels *ab* and *gh* are $A'B''H'$ and $A'G'H'$, respectively. These lines show that for maximum positive shear in panel *ab* and maximum negative shear in panel *gh* the uniform load must extend over the entire bridge.

In the construction of Fig. (*c*), the distance $A'H'$ $(=Np)$ is divided into $N-1$ equal parts by the lines $B'C''$, $C'D''$, etc.; hence $x_1=p/(N-1)$, $x_2=2p/(N-1)$, $x_3=3p/(N-1)$, or, in general, $x=np/(N-1)$, where n denotes the number of panels to the right of the panel in question. This is another independent derivation of Eq. (22).

The maximum stress, $S_{\max}$, in any diagonal is the product of $V_{\max}$ for the corresponding panel and the ratio h'/h. By laying off $A'A''$ as h'/h instead of unity, the values of $S_{\max}$ may be found directly. Thus $S_{\max}$ for the main diagonal in panel *cd*, for example, may then be found by multiplying the area of the triangle in the new figure, corresponding to triangle $QD''H$ in Fig. (*b*), by w.

It is apparent that under ordinary conditions this method is less convenient and expeditious than the numeric method, even without the aid of a slide rule. (See Example, Art. 124.)

(*b*) *Concentrated Loads.* The criterion for maximum shear in any panel from a concentrated load system may be derived from the influence line, Fig. (*b*), in the manner explained in Art. 190.

When the loading is in the position for maximum shear in panel *cd*, for example, there will be in general no loads to the left of *c*, so that only the two segments $C'D''$ and $D''H$ of the influence line are required. Therefore, as shown in Art. 190, if W_1 denotes the load in panel *cd*, W_2 the load for the distance a, and $W(=W_1+W_2)$ the total load on the bridge, the criterion for maximum shear in panel *cd* may be expressed as follows:

$$\frac{W_1}{x}=\frac{W_2}{a}=\frac{W}{a+x} \quad \ldots \ldots \quad (24)$$

$$W=\frac{a+x}{x}W_1,$$

in which $a = np$, and $x = np/(N-1)$. Substituting these values in the last equation and reducing,

$$W = NW_1,$$

which is the criterion for maximum shear in a truss with equal panels, and is seen to agree with Eq. (40), Art. 111.

The load system must be applied to the influence line in such a position as to satisfy the criterion for maximum for each panel in turn. This may be done by means of a load line (Art. 176) or a moment table (Art. 113). Finally, the maximum shears (or stresses) may be determined as the sums of the products of the individual loads and the corresponding ordinates to the influence line; or, better, as the sum of the products of the resultants of the loads for each segment of the influence line by the corresponding ordinates at the centers of gravity of these groups (Art. 182*a*).

For ordinary conditions the maximum live-load shears in a truss may be found more accurately and more rapidly by means of a moment table and slide rule than by influence lines. (Art. 111, and Example, Art. 115).

193. Influence Line for Bending Moment in a Truss. It has been shown (Art. 99) that the bending moments in a truss at the joints of the loaded chord are equal to those at corresponding sections of a beam; but that at intermediate sections the bending moments in a truss are less than at corresponding sections of a beam.

In the deck truss shown in Fig. 18 (*a*), the live-load stress in a member of the unloaded chord, as *bc*, for example, may be found by dividing the bending moment, M, at joint C in the loaded chord by the lever arm of *bc* with respect to C. The influence line for M, Fig. (*b*), may be drawn in precisely the same manner as for a beam, Fig. 14, Art. 188. Thus to any desired scale lay off the ordinates $AA' = a$ and $BB' = l - a$ from A and B respectively. Draw AB' and $A'B$, and as a check note that the point of their intersection, C', must lie vertically below C.

The criterion for ascertaining the position of loading for which M will be maximum is precisely the same as that for maximum bending moment at a given section in a beam, and it may be derived as in Art. 190. The value of M for any position of loading may be found by means of the influence line in the usual manner.

The live-load stress in a member of the loaded chord, as CD, for example, may be found by dividing the bending moment, M', at joint c in the unloaded chord by the height of the truss at that point. The influence line for M' is shown in Fig. (c). Since the bending moments at C and D are the same as those at corresponding sections in a beam, the segments AC' and $D'B$ are drawn in the same manner as in Fig. (b), the ordinates at A and B being now a_1 and $l-a_1$ respectively. The influence line is completed by the line $C'D'$ (Art. 191). The line $AQ''B$ is the influence line for the bending moment in a beam at the corresponding section z. The difference, $Q'Q''$, between the ordinates QQ'' and QQ' represents the bending moment at z in the track stringer in panel CD, from a unit load at z (Art. 99). The line $C'Q''D'$ is the influence line for the bending moment in the stringer, referred to $C'D'$ as a base.

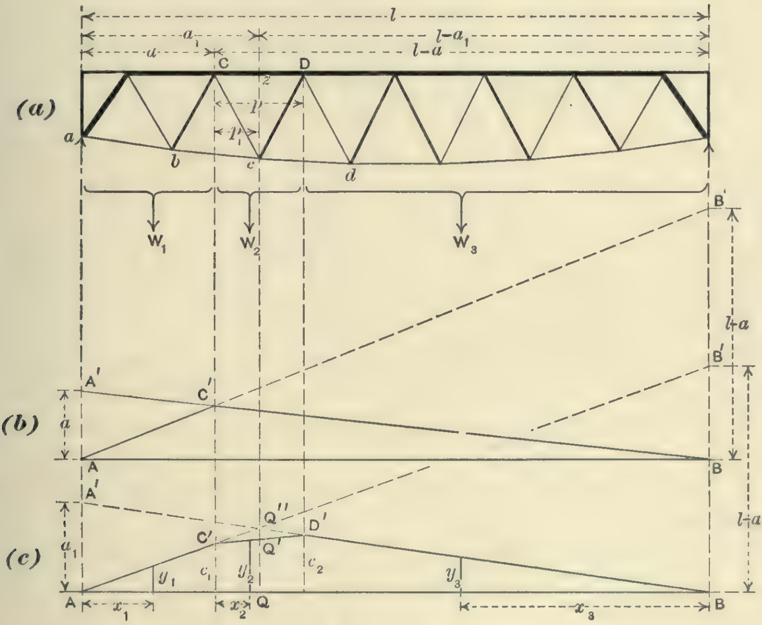

FIG. 18

The criterion for maximum M' may be derived from the influence line in a manner similar to that explained in Art. 190. Thus, if W_1, W_2, and W_3 denote the loadings for the distances a, p, and $l-(a+p)$, respectively.

$$M'=W_1y_1+W_2y_2+W_3y_3. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The slope of $C'D'$ is $(c_2-c_1)/p$ in which c_1 and c_2 are constants whose values may be found from the known slopes of AC' and $D'B$. Then y_1, y_2, and y_3 in Eq. (25) may be expressed in terms of x_1, x_2, and x_3, respectively, and known quantities, and the criterion for $M'_{\max}$ may then be found by differentiating the equation, and letting $dM'=0$. Finally, letting $W_1+W_2+W_3=W$, the resulting criterion will be

$$W=\frac{l}{a_1}\left(W_1+\frac{p_1}{p}W_2\right),$$

the same as Eq. (52), Art. 121. The present method is, however, so much less direct and simple than the algebraic method of Art. 121, that the operations outlined above will not be shown in detail. In general, influence lines cannot be used as advantageously as the algebraic method in the derivation of criteria for determining the position of loading for which the value of a given function will be maximum.

194. Influence Line for Web Stress in a Truss with Inclined Chords. Let it be required to find the criterion for maximum live-load stress in any web member in a truss with inclined chords, as, for example, member Cd in Fig, 19. It has been seen (Art. 138) that any load to the right of some point, q, in the corresponding panel will produce tension, and any load to the left of q will produce compression in that member; and that the stress in Cd from a load at q is zero. The point q may be readily located graphically by one of the following methods:

(*a*) *Position of Load for Zero Stress. First Method.* Produce CD to its intersections a' and h' with verticals at a and h. Draw $a'c$ and $h'd$, and produce these lines to their intersection q'. The desired point q will lie on the vertical through q'.

Proof. Let P_1 and P_2 be the joint loads at c and d from any load P at q. Let R and R_1 denote the end reactions from

P. The triangle $a'q'h'$ is then an equilibrium polygon for P, R, and R_1. Similarly, $cq'd$ is an equilibrium polygon for P, P_1 and P_2. In this polygon, P_1 and the stress in segment cd are the components of the stress in segment cq'. Similarly P_2 and the stress in segment cd are the components of the stress in segment dq'. The portion $cq'd$ of the equilibrium polygon $a'q'h'$, including P, may therefore be replaced by the segment cd and the loads P_1 and P_2. Thus $a'cdh'$ is an equilibrium polygon for P_1, P_2, R, and R_1. Taking the section 1, the intersection, G, of the segments $a'h'$ and cd, is a point on the vertical resultant of the intervening loads R and P_1 (Art. 19). But G is also the moment center of Cd. Therefore the stress in Cd from all the forces to the left of section 1 is zero, thus proving that the stress in Cd from a load P at q is zero.

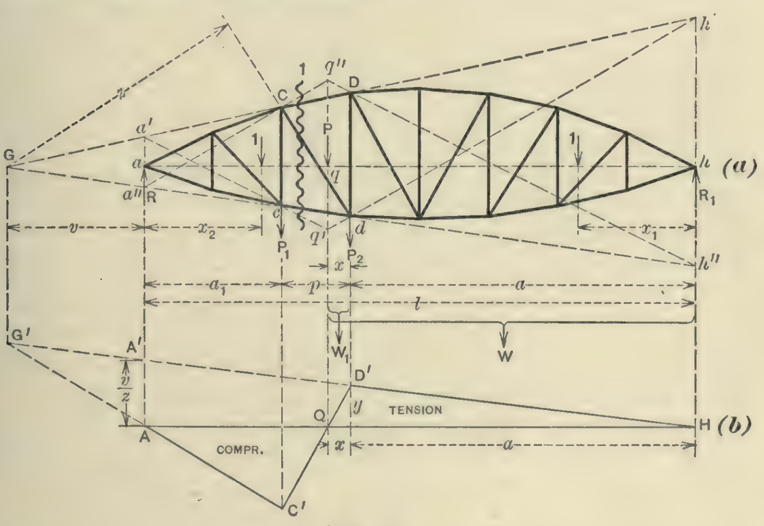

FIG. 19

Second Method. Produce cd to its intersections a'' and h'' with verticals at a and h. Draw $a''C$ and $h''D$, and produce these lines to their intersection q''. The desired point q will lie on the vertical through q'', as may be proven in the same manner as for the first method. The two methods are really identical in principle, and differ only in form of execution.

Third Method. This method, which is the simplest, will be explained at the end of this article.

(*b*) *Construction of Influence Line.* The influence line for stress Cd may now be constructed. For a load unity to the right of d, stress $Cd = R \times v/z = x_1/l \times v/z$, i.e., the stress varies directly with x_1, the distance of the load from the right support. The influence line for the distance a is therefore a straight line from H to a point D', Fig. (*b*), which may be located by computing stress Cd for a load unity at d, and laying off this stress as the ordinate y below apex d. The segment of the influence line for the panel length p must be a straight line (Art. 191), and since stress Cd for a load unity at q is zero, the point Q, vertically below q must be a point on the influence line. Therefore draw $D'Q$ and produce this line to its intersection C' with the ordinate at the other extremity of the panel. For a load unity to the left of c, stress $Cd = R_1 \times (l+v)/z = x_2/l \times (l+v)/z$, i.e., it varies directly with x_2, the distance of the load from the left support. The influence line for the distance a_1 is therefore a straight line connecting the point $C,'$ previously established, with A.

The line $D'H$ may also be drawn by laying off the ordinate AA' equal to v/z, and connecting A' and H. For the ordinate to HD' at any point is $x_1/l \times v/z$, and for a unit load at a, $x_1 = l$, and the ordinate equals v/z. Similarly, the ordinate at H to AC' produced (not shown) is $(l+v)/x$.

As a check on the accuracy of the drawing, note that $C'A$ and $D'H$, produced, should intersect on a vertical through the moment center, G. For it has been shown that the ordinate HH' (not shown) to AC' produced is $(l+v)/z$, and if $H'A$ is produced to its intersection G'' with $A'H$, the vertical intercepts at A and H of the triangle $HH'G$ are seen to be proportional to their respective horizontal distances, v and $l+v$, from G'. The fact that *the ordinates at the supports to the influence lines produced are proportional to the lever arms of the reactions with respect to the moment center used in obtaining the function represented by the influence line* is here proven for a moment center outside of the supports, and was proven in Arts. 188 and 193 for a moment center between the supports. This principle is thus seen to be perfectly general.

(*c*) *Criterion for Maximum Stress.* When the loading occupies the position for maximum stress in Cd, there will be in general no loads to the left of c, so that only the two segments $C'D'$ and

$D'H$ of the influence line will be required. Therefore , as shown in Arts. 190 and 192 (*b*), the desired criterion is

$$\frac{W_1}{x}=\frac{W}{a+x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The equation is seen to be precisely the same as the criterion for maximum shear in a truss, Eq. (24), Art. 192, although x has now a different value. In the present case x represents to scale the value expressed algebraically in Eq. (35), Art. 138. The algebraic value of x may also be derived from the known slopes of the influence line.

The position which will satisfy the criterion for maximum may be found either by means of a load line with wheel spaces laid off to the same scale as the distance scale for the influence line (Art. 176), or better, by scaling the distance x and employing a moment table.

(*d*) *Maximum Stress.* After the criterion is satisfied the value of the maximum stress may be found either by means of the influence line, or more rapidly and conveniently with the aid of a moment table and slide rule (Art. 136).

(*e*) *Maximum Counter Stress.* For maximum compression (counter stress) in Cd the left end of the truss is loaded, and only the two segments AC' and $C'D'$ of the influence line will be required. Letting $x'=p-x$, the criterion is then

$$\frac{W_1}{x'}=\frac{W}{a_1+x'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)a$$

in which, as before, W denotes the total load on the bridge and W_1 the load in panel *cd*.

*Third Method for Finding Position of Zero Stress.** The simplest general method for determining the position of the load for zero stress is as follows:

Referring to Fig. 20 (*a*), let a and h be the end supports of the truss, their relative vertical position being immaterial, and let cd be the panel in question. Required, x for zero stress in in Cd.

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* In Trans. Am. Soc. C. E., Vol. XLII (1899), p. 263, Mr. F. C. Kunz credits this method to Land, stating that it "applies to all cases of trusses with verticals and at least one horizontal chord." The horizontality of one chord is, however, not a necessary condition, as will be shown here by an extension of the proof given by Mr. Kunz.

Method. Let (1), (2), (3), (4), and (5) denote the verticals through G, a, c, d, and h, respectively. Produce CD to a' in (2), and cd to h' in (5). Then measure x horizontally from (4) to q, the intersection of $a'h'$ and Cd. If the lower chord ah is horizontal, h' lies at the right support. If the upper chord is also horizontal, aa' equals the height of truss.

Proof. Let the influence line for stress Cd, Fig. (b), be drawn in the most general way. Let the base line AH be given any

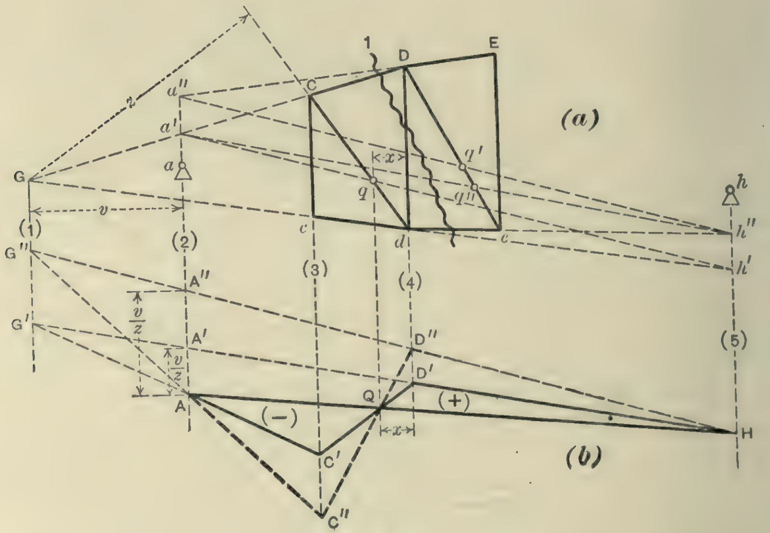


FIG. 20

slope. Select any point G' on (1). Draw $G'A$ and $G'H$. Produce $G'A$ to its intersection C' with (3) and draw $C'D'$. The line $AC'D'H$ is then an influence line for stress Cd. For it has been shown above that the segments $D'H$ and AC' produced must intersect at some point G' on (1). Since the location of G' depends only on the vertical scale of the ordinates to the influence line, G' may be *any* point on (1). Thus if G'' is chosen, the line $AC''D''H$ will be the influence line. Since in either case the construction of the influence line is based on correct principles, $C'D'$ and $C''D''$ must intersect the base line at a common point Q.

The scale of the ordinates to these influence lines is fixed by the fact that the ordinates AA' and AA'' at A are equal to v/z.

There remains now to show that q in Fig. (*a*) lies vertically above Q in Fig. (*b*). Since HA may have any slope and G' any position on (1), the two lines HG' and HA represent *any* pair of lines drawn from *any* point on (5) to *any two points* G' and A on (1) and (2). Hence, in general, by connecting these two points and producing the connecting line $G'A$ to its intersection C' with (3), then drawing $C'D'$ from C' to the intersection of HG' with (4), the point of intersection Q of HA and $C'D'$ must lie at a fixed horizontal distance, x, from (4).

The location of q in Fig. (*a*) conforms to this general principle. Thus the lines Gh' and $a'h'$ represent a pair of lines from a point on (5) to (1) and (2), respectively. Joining G and a' and producing Ga' to C on (3), then drawing Cd from C to the intersection of Gh' with (4), the point of intersection, q, of $a'h'$ and Cd must lie at the same horizontal distance, x, from (4) as Q in Fig. (*b*). Therefore q and Q must lie in the same vertical. The line $a'Cdh'$, referred to $a'h'$ as a base, is seen to be an influence line for stress Cd.

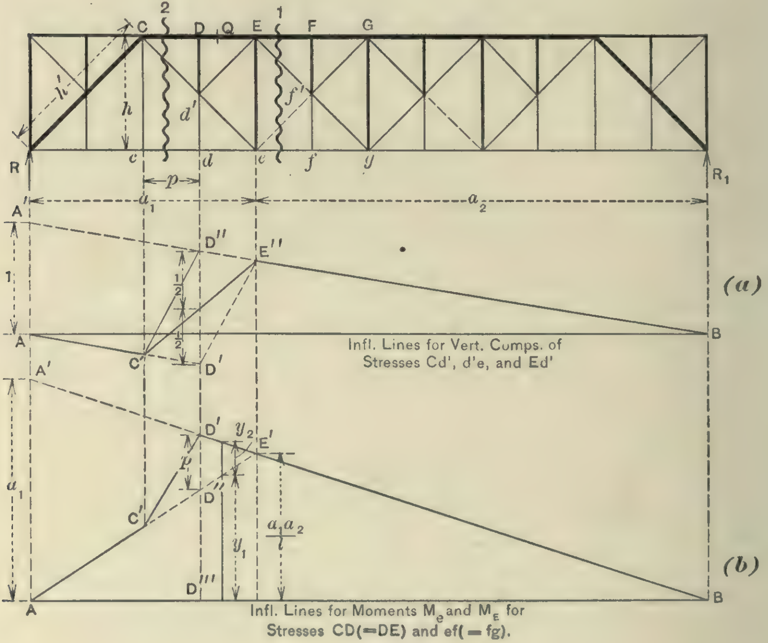
Similarly, to find the position for zero stress in De, produce DE to a'', and de to h''. The desired point, q', is the intersection of $a''h''$ and De.

Post Dd. If DE has the same inclination as CD, the load for zero stress in the diagonal De would lie in a vertical through q'' instead of q'. Stresses De and Dd would then become zero concurrently, as may be seen by analyzing the forces at D. But taking section 1, it is clear that stress Dd is not affected by the inclination of DE. Therefore a load in a vertical through q'' produces zero stress in Dd.

195. Influence Lines for Web Stresses in a Truss with Sub-Panels (Baltimore Truss). The construction of the influence lines for certain typical web members in the deck Baltimore truss shown in Fig. 21 will be briefly considered.

Stresses Cd', $d'e$, and Ed'. In Fig. (*a*), $AC'E''B$ would be the influence line for the shear in the double panel CE if the secondary web system did not exist (Art. 192). Similarly, $AC'D''B$ is the influence line for the shear in panel CD, and $AD'E''B$ is the influence line for the shear in panel DE. These lines depend evidently only on the panel divisions of the truss, irrespective of whether these divisions are main panels or sub-panels.

If the end ordinate AA' is laid off as h'/h instead of 1, $AC'D''B$ will be the influence line for stress Cd'. Again, $C'D'E''$ is then the influence line, with reference to $C'E''$ as a base line, for stress Ed'. That member is affected only by loads in the double panel CE. For a load unity at D the vertical component of stress Ed' is $\frac{1}{2}$, and as the load moves from D in either direction, the joint load at D varies uniformly from unity to zero. The line $AD'E''B$ is now the influence line for the shear in panel DE multiplied by h'/h; and therefore for the difference of stresses $d'e$ and Ed', which are both tensile. Combining this influence line with $C'D'E''$, the resultant $AC'E''B$ is the influence line for stress $d'e$. This is seen to be same as in a truss without sub-panels, as it should be, since any load at D is transmitted to C and E without producing any stress in $d'e$, just as in the case of a truss without sub-panels.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a2
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E''
1
1/2
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A
C'
D'
B
(a)
Infl. Lines for Vert. Comps. of Stresses Cd', d'e, and Ed'
A'
D'
y2
E'
p
D
a1
C'
a1a2/l
y1
D'''
(b)
A
B
Infl. Lines for Moments Me and ME for Stresses CD(=DE) and ef(=fg).

FIG. 21

196. Influence Lines for Chord Stresses in a Truss with Sub-Panels. The analysis of the chord stresses in the Baltimore truss, Fig. 21, is equally applicable to the Pettit truss, except that the lever arm by which the resultant moment is divided is affected by the inclination of the chord.

Stress $ef(=fg)$. Referring to Fig. 21, the counter ef' is not in action when the loading is in position for maximum stress in any chord member. Taking section 1, the moment center of stress $ef(=fg)$ is seen to lie at E, and that stress is equal to M_E, the bending moment at E, divided by h, the height of truss. The bending moment M_E in the truss is precisely the same as that at the corresponding section of a beam. The influence line $AE'B$ for M_E, Fig. (*b*), may therefore be drawn as in Fig. 14, Art. 188, the ordinate at e having the same value, a_1a_2/l. To draw BE', lay off $AA'=a_1$, and to draw AE', lay off an ordinate BB' (not shown) equal to a_2, upward at B. As a check, note that E', the intersection of $A'B$ and AB', must lie in a vertical through the moment center E.

Stress $CD(=DE)$. Taking section 2, the moment center of stress $CD(=DE)$ is seen to lie at e. Considering the resultant moment, M_e, of the forces to the left of the section it is seen that for loads in the double panel CE, M_e is *larger* than the bending moment at $e(=M_E)$, because the joint load at D lies to the *right* of the section, and its moment, m, about e is therefore not subtracted from the moment, Ra_1, of the reaction. Thus, considering the values of M_E and M_e for a load unity at D, for which $m=1\times p$,

$$M_e=M_E+p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If the sub-paneling in the double panel CE did not exist, M_e and M_E would be equal and $AE'B$ would also be the influence line for M_e. But the sub-paneling cannot affect conditions to the left of C or to the right of E. Therefore AC' and BE' form part of the influence line for M_e. For the unit load in any position from the right end of the truss to D, the moment at at e is Ra_1. Hence the segment BE' of the influence line must be continued to D', and since the influence line for a single panel is straight (Art. 191), $D'C'$ will complete the influence line for M_e.

For a load unity at D, $\overline{D'D'''}=M_e$ and $\overline{D''D'''}=M_E$. Therefore, by Eq. (27), $\overline{D'D''}=p$. This may also be proved by the similar triangles $AA'E'$ and $D''D'E'$. Since AA' is equal to the horizontal distance from that ordinate to E', that is true also of $D'D''$; therefore $\overline{D'D''}=p$.

If S denotes the stress CE for a load unity at any point, Q, in the double panel, then from Fig. (*b*),

$$S=\frac{y_1}{h}+\frac{y_2}{h}, \qquad (28)$$

in which y_1/h is the stress in CE as a member of the primary truss system, and y_2/h, its stress as a member of the secondary truss $Cd'E$. This truss is shown separately in Fig. 22(*a*). The influence line for the bending moment at the center of the truss is shown in Fig. (*b*) with reference to a horizontal base line, and in Fig. (*c*) with reference to an inclined base line. In both figures $D'D''=\frac{1}{2}p$. If S_1 denotes the stress CE for a load unity in any position, Q, $S_1=y/\frac{1}{2}h$. But from Eq. (28), $S_1=y_2/h$, which can be true only if $y_2=2y$, for corresponding positions in the two figures. A comparison of the ordinates $D'D''$ shows that condition to be satisfied. The line $C'D'E'$ in Fig. 20 (*b*) is thus seen to be the influence line for *twice* the bending moment at the center of the secondary truss $Cd'E$.

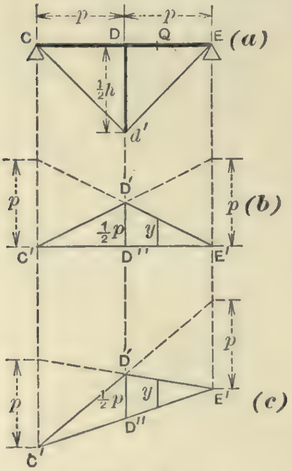


FIG. 22

197. Position of Loading for Maximum Value of a Function. General Method. If the conditions are somewhat complex, as, for example, for skew bridges (Chapter XII), swing bridges, arches, etc., the criterion for maximum cannot be reduced to a simple expression. In such cases the maximum value of the function may be found with the aid of the influence line by Winkler's Method to be explained presently, with due regard to certain general principles to be first considered.

(*a*) *Single Load.* Suppose that the line $AB'C'D'E'F$, Fig. 23 (*a*), is the influence line of some function, K. Assume a load P applied at any point from A to F. The positive value of the function will evidently be maximum when P is at C, and its value is then $K_{\text{max.}} = Pa$. To either side of C the ordinate to the influence line is less than a. This can be true only if the segments of the influence line on both sides of C' lie between the base line and a line through C' parallel to the base line.

The same condition is met at E' for maximum negative value of the function.

As the load P moves along AF, the function K will have a constant rate of variation for the space commanded by each segment of the influence line, but the rate of variation changes abruptly as the load passes from one of these spaces to the next. Thus for AB, the rate of variation of the function is $P \tan \alpha_1$; for BC, $P \tan \alpha_2$; for CD, $P \tan \alpha_3$, etc. As the load P moves from right to left, the differential change, dK, of the function for a differential movement dx just to the right of C is $dK = +Pdx \tan \alpha_3$, whereas for a like movement just to the left of C, $dK = -Pdx \tan \alpha_2$. For the former the rate of change of the function $dK/dx (= +P \tan \alpha_3)$ is positive, while for the latter, $dK/dx (= -P \tan \alpha_2)$ is negative, thus showing mathematically that the function is maximum when P is at C. For a movement across C in the opposite direction dK/dx also changes from positive to negative. Similarly, the rate of variation of the negative value of the function changes from positive to negative as the load moves across E in *either* direction, thus indicating that $-K$ is maximum when the load is at E.

(*b*) *Series of Loads.* Considering now the general case of a series of concentrated loads for which $K = \Sigma Wy$, it is evident that the rate of variation of K will be constant until a load passes any one of the division points A, B, C, etc., when the rate of variation will be subject to an abrupt change. Hence

I. The function can be maximum (or minimum) only when one of the loads is at an ordinate to one of the angles of the influence line.

Let angles between adjacent segments of the influence line on the side *toward* the base line be called *concave* angles if less than 180°, as at C', D', and E', and *convex* angles if greater than 180°, as at B'. It will now be shown that

II. The function can be maximum only when one of the loads is at an ordinate to a *concave* angle of the influence line, and minimum only when one of the loads is at an ordinate to a *convex* angle of the influence line.

Suppose, for example, that the load system occupies some position for which a load w is at a differential distance dx to the right of D. Let W_1, W_2, W_3, etc., denote the resultants of the loads for the distances AB, BC, CD, etc. The differential change in K for a movement dx toward the left is then

$$dK = -W_1\Delta_1 - W_2\Delta_2 + W_3\Delta_3 + W_4\Delta_4 - W_5\Delta_5,$$

whence

$$\frac{dK}{dx} = -W_1 \tan\alpha_1 - W_2 \tan\alpha_2 + W_3 \tan\alpha_3 + W_4 \tan\alpha_4 - W_5 \tan\alpha_5. \quad . \quad (29)$$

Suppose that this value of dK/dx is positive, and that the system is moved another distance dx toward the left so that in Eq. (29) w will now be included with W_3 instead of W_4. For a *concave* angle, $w \tan\alpha_3 < w \tan\alpha_4$, (if the direction of $D'C'$ is downward from D', $w \tan\alpha_3$ becomes negative), and the value of dK/dx is then abruptly decreased. Suppose that dK/dx is now negative. The rate of variation of the function will then have changed from a positive to a negative value in a differential movement, and will have attained a *maximum* value when w is at D. If the angle at D' is *convex*, $\tan\alpha_3 > \tan\alpha_4$, so that then dK/dx cannot change from a positive to a negative value.

If, on the other hand, dK/dx by Eq. (29) is negative, then for a *convex* angle it will be abruptly *increased* as w passes D, and if its value is then *positive*, K will have reached a *minimum* value when w is at D.

Hence, in general, to find the position for maximum proceed as follows:

Apply the space diagram to the base line in a position with the heaviest loads grouped in the region where the ordinates are maximum, and with some critical load at the ordinate to a *concave* angle of the influence line. Then proceed by one of the following methods:

Numeric Method. Set up an equation similar to Eq. (29) if the rate of change is to be expressed for a differential movement toward the left; or with all signs in the second member of the equation reversed if the movement is in the opposite direction. In the former case include the critical load with the loading, first for the segment to the right, and second, for the segment to the left. In the latter case, reverse this order. In either case, if dK/dx is first positive and then negative, the criterion for maximum is satisfied.

Graphic Methoa (*Winkler's Method*). Referring to Fig. 23(*c*), lay off the loads 1, 2, 3, etc., in the same sequence as in Fig. (*b*). The spaces W_1, W_2, W_3, etc., are then proportional to the loadings in Fig. (*b*) for the distances commanded by the successive segments of the influence line. Load 4 at C is the critical load. Let dK/dx be found graphically with load 4 first to the right, and second, to the left, of C. Since this indicates a movement toward the left, begin at f, Fig. (*c*), and draw fe, ed, dc', etc., parallel to FE', $E'D'$, $D'C'$, etc., respectively, in Fig. (*a*). Then evidently the vertical projections of fe, ed, etc., are equal to $-W_5 \tan \alpha_5$, $+W_4 \tan \alpha_4$, etc., respectively, and aa' represents their algebraic sum. Therefore, $aa' = dK/dx$ when load 4 is to the right of C. Similarly

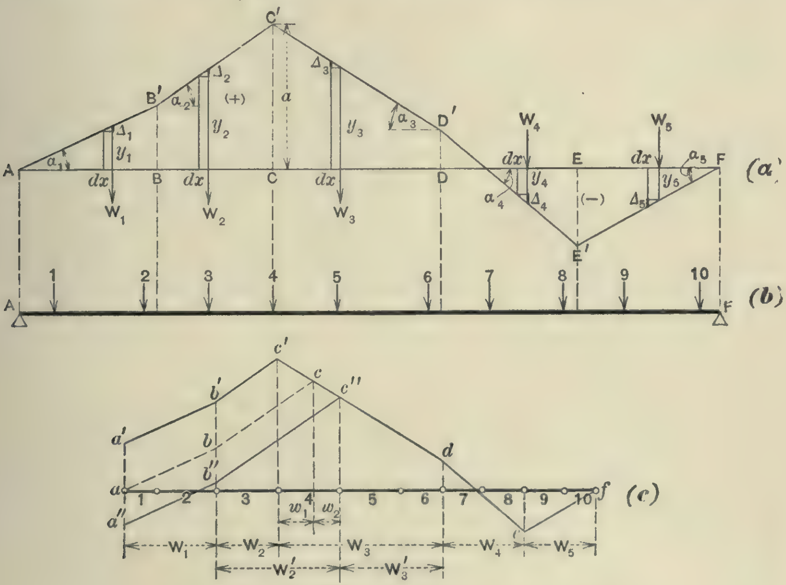


FIG. 23

$aa''=dK/dx$ when load 4 is to the left of C. Since aa' is above the base line, or positive, and aa'' is below the base line, or negative, the criterion for maximum is fulfilled when load 4 is at C. Instead of drawing both $c'b'a'$ and $c''b''a''$, the first line $fedc'$ may be stopped at c' and a second line abc may be drawn from a. If the point of intersection, c, of these two lines lies above space 4, the criterion is satisfied. Mathematically, if 4 is then supposed to be divided into two parts, w_1 and w_2, Fig. (*c*), of which w_1 acts just to the left of C, and w_2 just to the right of C, $dK/dx=0$.

In the foregoing it has been assumed that the loading is distributed over the entire distance AF. It is evident, however, from Fig. (*a*) that the function will attain its greatest positive value when the loading extends only over the distance for which the influence line lies above the base line.

Note. Referring to Fig. 20 (*b*), Art. 195, $AC'D'B$ is the influence line for moment M_e for chord stress $CD(=DE)$. From the above discussion it is seen that the critical load for maximum M_e must be placed at D, the middle of the double panel, since the angle $C'D'E$ is a *concave* angle, and that it must not be placed at C or E. Since the same influence line applies to a Pettit truss, the critical loads in Figs. 14 and 15, Art. 166, and Fig. 16. Art. 167, were placed at the middle of the double panel.

198. Influence Table. If the stress in every member of a truss is determined for a load unity at each joint of the loaded chord in turn, the stresses from any other loading may be found by multiplying each set of stresses for a unit load at a given joint by the actual load at that joint, and by summing these products for each member. This procedure is especially well adapted to the determination of the stresses from an 'equivalent' uniform loading by the conventional method (Art. 98),

INFLUENCE TABLE

Member.	Stresses for Load Unity at						Σ Stresses.	
	A	*B*	*C*	*D*	*E*	*F*	+	−
ab								
bc								
etc.								

in framed structures such as arches, and swing bridges whose analysis is somewhat complex, or for which the criterion for maximum stress cannot be reduced to a simple expression. In such cases it is often convenient to find the stresses graphically by constructing a separate diagram for the load unity at each joint. The resulting values may then be collected in the form of an *influence table,* arranged as illustrated above.

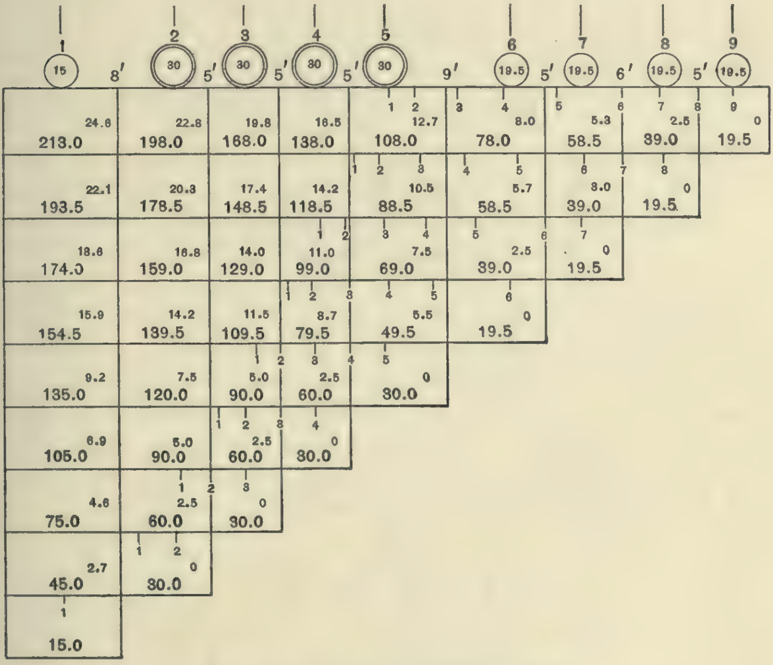
The quantities in the first horizontal line of this table, for example, are then the values of the ordinates to an imaginary influence line for stress *ab*. The sums of the plus and minus quantities in each horizontal line are entered in the last two columns. If w denotes the intensity, per truss, of the uniform load, and p the panel length, the maximum stresses are found by multiplying the joint load wp by each of the values in the last two columns. In case the truss contains counters, the stresses whose value may be maximum when either the counter or the main diagonal in a given panel is in action, should be found for both conditions and combined with the dead-load stresses for corresponding conditions.

An influence table is sometimes convenient not only for maximum stresses, but for simultaneous stresses in general. Thus in deflection and camber computations it is necessary, as will be seen later (Chapter XVI), to find the simultaneous stresses for a given condition of loading—usually a uniform load covering the entire bridge. The simultaneous stresses for the latter may be found by multiplying the algebraic sums of the values in the last two columns by wp. Again, in the proportioning of pins it is necessary, as will be seen later (Vol. III), to ascertain the simultaneous stresses in the members meeting at a given joint under loading which will produce maximum stress in one or more of those members. For uniform loading when treated by the conventional method (Art. 98), these stresses may be found by combining, for each member, those values in the influence table which are in the same columns as those used to find the stresses which are maximum. These combined values must then be multiplied by wp.

199. Use of Influence Lines in Comparison with Other Methods. Practical Hints. In regard to the generality of its applicability, and the completeness of the solutions to which it leads, the influence line method has no equal. If the influence line of a desired function is once constructed, the problem is solved, and the variation of the function with the position of the load is shown in an objective and complete manner. The value of the function for any loading in any position may then be found by the simple process of scaling certain ordinates to the influence line, and finding the algebraic sum of the products of these ordinates and the corresponding loads. The influence line is therefore invaluable in the solution of complex problems in stresses, deflections, etc.

For the derivation of criteria for maximum stresses in the form in which these criteria must be used in the ordinary methods of computation, the application of the influence line is, however, less direct and convenient than the algebraic method employed in Chapters IV, VII, VIII, and X. Similarly, for the determination of stresses in ordinary bridge trusses, the algebraic and graphic methods treated in earlier chapters afford more rapid and convenient means than the influence line. Influence lines are useful, however, in finding the position for maximum stress in the web members of trusses with inclined chords. (Art. 194.)

In the application of influence lines to more complex analyses the final operations may be facilitated in various ways. Thus a *center of gravity table* may be prepared in a form similar to that of the moment table, as shown for a single engine of Cooper's E–60 Loading in Fig. 24. The principal figures in the table represent the sums of any series of consecutive loads. For example, to find the sum of loads 3 to 6, inclusive, follow the column under wheel 6 downward to its extremity, continue to the left, and read the value, 109.5, in the column under wheel 3. The upper figure, 11.5, in the same space, denotes the distance of the center of gravity of loads 3 to 6, inclusive, from wheel 6, and the position of the center of gravity is indicated by the point marked 3 on the horizontal line above this space. Similarly, to locate the center of gravity of wheels 2 to 7, inclusive, follow the column under wheel 7 to the lowest space, and continue to the left along the line above this space until the point marked 2 is reached. In using the table it will be convenient to transfer these points permanently to the upper edge by means of hair lines. By applying the upper edge of such a table to the base line of an influence line drawn to the same horizontal scale, the center of gravity of the loads for each segment may be readily located, and the ordinate to the influence line at that point may either be scaled or computed and then multiplied by the sum of the loads, obtained from the center of gravity table.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19.5
19.5
19.5
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8′
5′
5′
5′
9′
5′
6′
5′

24.6 213.0	22.8 198.0	19.8 168.0	16.5 138.0	12.7 108.0	8.0 78.0	5.3 58.5	2.5 39.0	0 19.5
22.1 193.5	20.3 178.5	17.4 148.5	14.2 118.5	10.5 88.5	5.7 58.5	3.0 39.0	0 19.5	
18.6 174.0	16.8 159.0	14.0 129.0	11.0 99.0	7.5 69.0	2.5 39.0	0 19.5		
15.9 154.5	14.2 139.5	11.5 109.5	8.7 79.5	5.5 49.5	0 19.5			
9.2 135.0	7.5 120.0	5.0 90.0	2.5 60.0	0 30.0				
6.9 105.0	5.0 90.0	2.5 60.0	0 30.0					
4.6 75.0	2.5 60.0	0 30.0						
2.7 45.0	0 30.0							
15.0								

FIG. 24.—Center of Gravity Table.

Again, the ordinates to the influence line at equal loads may be mechanically added by means of a pair of dividers of suitable size, or by means of a 'rider' on the scale by which the vertical ordinates are measured.

It is to be understood, finally, that the load unity frequently referred to in this chapter may be laid off to represent any convenient unit, such as 10, 100, 1000, 10,000, etc., the other ordinates being then scaled as fractional parts of that unit.

CHAPTER XII

SKEW BRIDGES AND BRIDGES ON CURVES

A. SKEW BRIDGES

200. General Considerations. When the alinement of a crossing is oblique to the direction of the stream, or, in the case of overhead city crossings, to the direction of the street, recourse must usually be had to skew bridges. In the former case, the piers must be built with a view of offering least obstruction to the flow of the stream and to navigation; and, in the latter, the faces of the abutments must, for obvious reasons, be parallel to the street lines below. Unless local conditions admit of no better alternative, the use of skew bridges should be avoided. Compared with square bridges, they are more expensive, both as to masonry and superstructure, less rigid and less sightly.

The computation of the stresses in skew bridges for concentrated load systems, though somewhat tedious, presents no special difficulties. The formulas for bending moments and shears applicable to the most general cases may be much simplified for the conditions usually met in practice and, with the aid of a moment table for the particular load system employed, the labor of computation may be greatly reduced.

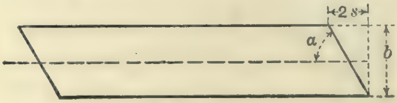


FIG. 1.

Referring to Fig. 1, the angle α between the center line of the bridge and a line joining the supports at one end is called the *skew angle*. In the limiting case, i.e., a square bridge, this angle becomes 90°.

The distance $2s$ is called the *skew*. If b is the width of the bridge, center to center of girders or trusses,

$$2s=\frac{b}{\tan\alpha}. \qquad (1)$$

Skews are distinguished as 'right-forward' (Fig. 1) or 'left-forward' (Fig. 2), according to whether the right or left truss is farther removed from an observer approaching the bridge from the end in question.

201. General Design. In skew bridges, as frequently designed, the panels are made of equal lengths on the center line of the bridge, as shown in Fig. 2, which represents the general, though uncommon, case of unequal skews. For a through bridge, the inclination of the end posts is made equal to the panel length. This lends a symmetrical outline to the truss as a whole, and causes the portals between the end posts to lie in plane instead of warped surfaces. In the case of a Pratt truss, here shown, the intermediate posts are all vertical, so that the transverse floor beams, which are at right angles to the planes of the trusses, may be riveted directly to these posts. The only irregularities appear in the inclination of the end suspenders, and in the length of the end chord members. In the usual case of equal skews these end suspenders, though inclined, are parallel, and the individual trusses differ little in their general appearance from the trusses in a square bridge. If the skew equals the panel length, the floor system may be advantageously arranged as

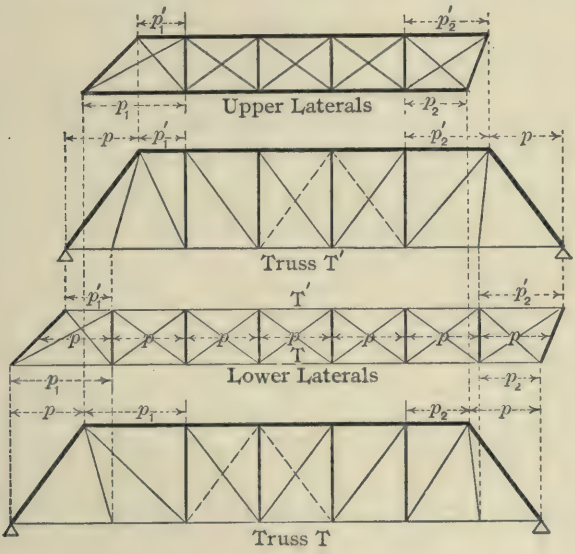


FIG. 2.

shown in Fig. 3, whereby all irregularities in the main trusses disappear. It is desirable to avoid the use of inclined end suspenders even where the skew does not equal the panel length. This may be accomplished as follows:

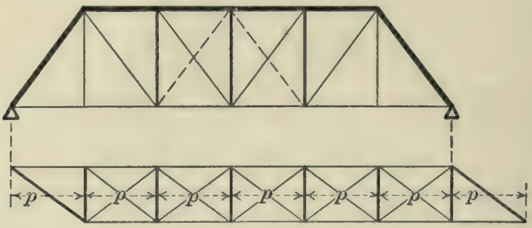
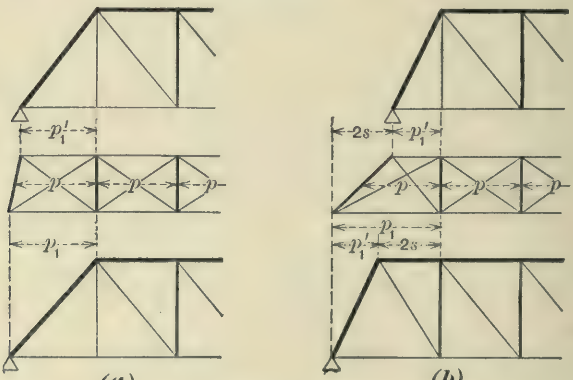


Fig 3.

For small skews the inclined end posts may be given unequal slopes, as in Fig. 4 (*a*). In order that the portals may lie in plane surfaces, as is desirable, their inclination may be a mean between those of the end posts, and their connections to the latter may be designed accordingly.



(*a*) (*b*)

Fig. 4.

For larger skews the trusses may be arranged as in Fig. (*b*), in which the end posts have the same slope and the suspender is omitted at one end.

202. Assumed Distribution of Loading. In the analysis of the live-load stresses in the trusses of a skew bridge the computations may be greatly simplified by assuming each floor beam to be symmetrically loaded, so that the load on each floor beam is communicated in equal parts to the two trusses. So far as the trusses are concerned, this is equivalent to assuming the live load to be applied along the center line, XX (Fig. 5), of the bridge. In reality, the first floor beam, bb', from each end of the bridge, supports end stringers, t and t', of unequal lengths, so that this floor beam is not loaded symmetrically. In designing this floor beam and in determining the vertical components of the maximum live-load stresses in the inclined end suspenders, Fig. 2, which support the floor beams at b and b', Fig. 5, the actual conditions should be considered. In the analysis of the live-load stresses in all other truss members, however, the assumption that floor beam bb' is symmetrically loaded is sufficiently accurate for practical purposes and is commonly made in practice.

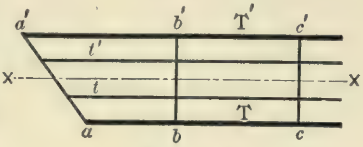


Fig. 5.

With a view of avoiding unnecessary irregularity in the lengths of stringers the panel divisions may be arranged as in Fig. 6, by which the length of stringer t' is made equal to that of the intermediate panels. In the best practice the end stringers t and t' are made equal as shown in the figure, so that both ends of the cross-ties near the ends of the bridge may be supported by the stringers, instead of one end resting on a stringer and the other on the masonry. An added advantage is that the end stringers will then be alike and will deflect equally. For this arrangement the assumption that the live load is applied along the center line involves no error.

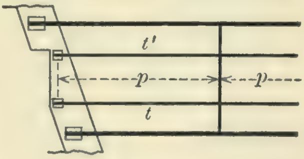


Fig. 6.

203. Position of Concentrated Load Systems for Maximum Stress. In the analysis of live-load stresses in the trusses of skew bridges from concentrated load systems it is assumed, as has been stated, that the loading is applied along the center line of the bridge. This assumption, though necessary for the derivation of convenient formulas and sufficiently accurate for practical purposes, does not warrant the use of refined methods for determining the exact position of this more or less fictitious loading for maximum stress in each member. While for maximum web stresses comparatively simple criteria can be derived, the criteria for maximum chord stresses are more complicated, and for reasons stated such criteria are of little or no practical value. The position of the loading for maximum stress may be found with sufficient accuracy by applying the criteria derived for square bridges. Since the chords in skew bridges are with rare exceptions horizontal, Eq. (39), Art. 111, may be used in general as the criterion for maximum web stress, and Eq. (42), Art. 111, for maximum chord stress. These criteria are to be interpreted in precisely the same way as in in the case of a square bridge having the same number of panels as the skew bridge, and a length equal to the length of the skew bridge along the center line.

204. General Methods of Stress Analysis. *Dead-load Stresses.* The analysis of the dead-load stresses in skew bridges presents no new features. After the apex loads have been determined, the web stresses may be conveniently found by the method of shears (Art. 58), and the chord stresses by the summation of the horizontal stress increments at successive joints (Art. 59) beginning at each end and proceeding toward the center, where the agreement of the stress in the middle chord members will serve as a check. Since the trusses of skew bridges are unsymmetrical, and the loads at the first apexes from the ends in both chords differ from the loads at the intermediate apexes, the computations cannot be performed as quickly as in the case of square bridges.

Stresses from Uniform Live Loading. For uniform live loading the web stresses in skew bridges should be based on the conventional method of loading (Art. 98). The web stresses may be conveniently found by the methods recommended above for dead-load stresses, and the chord stresses by the method of moments (Art. 57). For skew bridges with inclined suspenders the ratio between the dead and live-load chord stresses is not constant, as for square bridges, because in the former the dead loads at the hip joints are not vertically above the dead and live loads at the lower extremities of the suspenders.

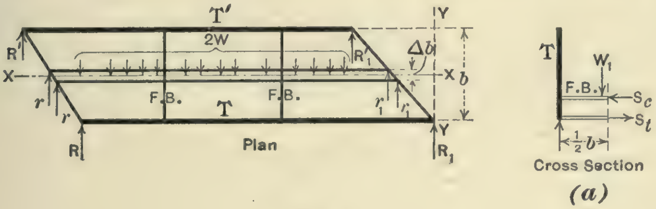
Stresses from Concentrated Live Loads. The live-load stresses from concentrated load systems may be found by two general methods, which may be distinguished as the *direct* and *indirect* methods.

(*a*) *Direct Method.* By the direct method the reaction at the end of the truss from the load system in any position is found directly by means of special formulas to be derived presently, which may be evaluated with the aid of a moment table.

(*b*) *Indirect Method.* By the indirect method the floor-beam loads throughout the bridge are first determined for the load system in the position for the maximum stress in each member. These operations may be conveniently performed with the aid of a moment table, as explained in Art. 46. The floor-beam loads are assumed to be communicated in equal parts to the two trusses. The web stresses may then be found by the method of shears (Art. 58) and the chord stresses by the method of moments (Art. 57).

The indirect method requires no further explanation. In its application it is more laborious than the direct method.

205. Truss Reactions from Concentrated Live Loads by Direct Method. A skew bridge is shown in plan in Fig. 7. The directions of the loads and reactions are in reality normal to the plane of the paper, but for the sake of clearness they are represented as shown in the figure. The assumption that the live load is applied along the center line, XX, of the bridge is equivalent to assuming the load supported by a pair of stringers a very small distance, Δb, apart. The reactions r and r_1 of the end stringers at the same end of the bridge are then equal. Suppose, now, that the entire bridge is cut into two parts by the section XX, the total live load, $2W$, being applied in equal parts to the stringers on either side of the section. Since the loading on each floor beam, F.B., is counterbalanced, the shears at the middle of the floor beams are zero. The only stresses at this section of a floor beam are then the resultants, S_c and S_t, Fig. (*a*), of the compressive and tensile flange stresses. The vertical



(*a*)

FIG. 7.

load, W_1, on the floor beam is held in equilibrium by these forces and a reaction equal and opposite to W_1, at the truss. Considering that part of the bridge below XX, its load, W, is held in equilibrium by the forces R, R_1, r, r_1 and the stresses S_c and S_t in the floor beams. If YY is an axis in a horizontal plane in a direction perpendicular to T the moments of the stresses S_c and S_t, which are parallel to YY, about that axis are zero. For equilibrium the resultant moment of the remaining forces W, R, R_1, r, and r_1 about YY must therefore be zero.

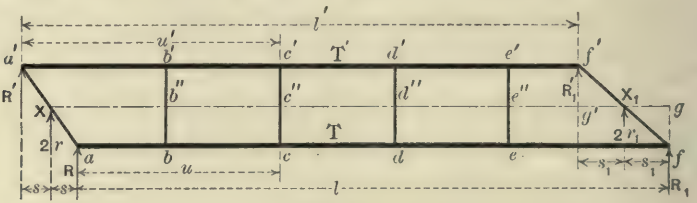


FIG. 8.

Assuming now the general case of unequal skews, $2s$ and $2s_1$, shown in plan in Fig. 8, let the live load, $2W$ (not shown), be applied along the center line XX_1, and consider the moments about axis fg of the forces acting on that part of the bridge below XX_1. If M_g denotes the moment of W, the load on one rail, about g,

$$Rl+r(l+s)+r_1s_1-M_g=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, by moments about axis $f'g'$ of the forces acting on that part of the bridge above XX_1, letting $M_{g'}$ denote the moment of W about g',

$$R'l'+r(l'-s)-r_1s_1-M_{g'}=0. \quad . \quad . \quad . \quad . \quad (3)$$

From Eqs. (2) and (3),

$$R=\frac{M_g-rs-r_1s_1}{l}-r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$R'=\frac{M_{g'}+rs+r_1s_1}{l'}-r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When rs and r_1s_1 are small in comparison with M_g and $M_{g'}$, as is usually the case, these formulas may be reduced, with slight errors, to the following forms:

$$R=M_g/l-r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)a$$

$$R'=M_{g'}/l'-r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)a$$

The reactions R_1 and R_1' at the other end of the bridge will be maximum for the engines advancing on the bridge from the left. Since the moment table is designed, however, for the engines headed in the opposite direction, it is more convenient to consider the plan in Fig. 8 to be rotated through 180° in the plane of the paper, causing f and f' to assume the positions of a' and a respectively. Reaction R_1' may then be found by Eq. (4), and reaction R_1 by Eq. (5) after substituting s for s_1, s_1 for s, l for l', and l' for l.

For the usual case of equal skews, reactions R and R' suffice. The stresses in the left half of truss T, for the load headed toward the left, are then equal to those in the right half of truss T' for the load headed toward the right. In a similar sense the stresses in the left half of truss T' are then equal to those in the right half of truss T.

If the stringers in the end panel are of equal length, as in Fig. 6, Art. 202, the equations for the reactions are unchanged, but s and s_1 denote the distances, in the direction of the axis of the bridge, from the supports of the truss in question to the supports of the stringers.

The values M_g, $M_{g'}$, r, and r_1 may be readily found by the aid of a moment table. If M_{X_1} denotes the moment of W about X_1, $M_g = M_{X_1} + Ws_1$ and $M_{g'} = M_{X_1} - Ws_1$.

206. Live-load Stresses. (*a*) *Web Members.* After the truss reactions have been found by the method explained in the preceding article, the corresponding shears and web stresses may be found in the usual manner (Art. 58).

For maximum shears in panels ab and $a'b'$, the critical load is placed at b'', and the loading extends into the left end panel. The reactions R and R' by Eqs. (4) and (5) are then the shears in panels ab and $a'b'$, respectively.

For maximum shears in panels bc and $b'c'$, the critical weight is placed at c''. When the criterion (Eq. (39), Art. 111), is satisfied, no loading exists in the left-end panel; hence in Eqs. (4) and (5) $r=0$, and these equations become

$$R = \frac{M_g - r_1 s_1}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$R' = \frac{M_{g'} + r_1 s_1}{l'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Similarly, from Eqs. (4)a and (5)a,

$$R = M_g/l, \qquad (6)a$$

$$R' = M_{g'}/l'. \qquad (7)a$$

The shears in the intermediate panels are found in the usual manner by subtracting the forward joint loads from the values of the reactions given by Eqs. (6) and (7). The stress in any diagonal is obtained finally by multiplying the shear in the corresponding panel by the secant of the angle between that diagonal and a vertical.

(*b*) *Chord Members.* The live-load chord stresses may be conveniently found by the method of moments (Art. 57). For maximum chord stresses the live load must be distributed over the entire bridge. Since the end panels are therefore always occupied, the reactions for the loadings which satisfy the criteria for maximum stress must be found by Eqs. (4) and (5).

Referring to Fig. 8, Art. 205, let it be required, for example, to find the maximum bending moments, M and M', at c and c', respectively. The critical weight must be placed at c'', and the position found which will satisfy the criterion (Eq. (42), Art. 111). Let m denote the moment of the loads (on *one* rail) to the left of c'' about that point. Then

$$M = Ru + r(u+s) - m. \qquad (8)$$

Similarly, for maximum bending moment, M', at c'.

$$M' = R'u' + r(u'-s) - m. \qquad (9)$$

Substituting in these equations the values of R and R' from Eqs. (4) and (5),

$$M = (M_g - rs - r_1s_1)u/l + rs - m, \qquad (10)$$

$$M' = (M_{g'} + rs + r_1s_1)u'/l' - rs - m. \qquad (11)$$

For moderate skews these equations may be changed to the following simplified forms without much error:

$$M = M_g u/l - m, \qquad (10)a$$

$$M' = M_{g'} u'/l' - m. \qquad (11)a$$

For the usual case of equal skews the bending moments at the joints in the left half of truss T may be found by Eq. (10)

or (10)a, and those in the left half of truss T' by Eqs. (11) or (11)a. The former apply then to the right half of truss T', and the latter to the right half of truss T, for the load system headed in the opposite direction. The chord stresses are finally found in the usual manner by dividing the bending moment at the moment center of each member by the height of the truss.

The expressions for reactions and bending moments in the last three articles were derived for a right-forward skew bridge. For a left-forward bridge the same formulas may be used, observing only that the letters T, R, l, u, etc., always refer to the truss which meets the *left* end strut aa' in an *obtuse* angle, while T', R', l', u', etc., refer to the truss which meets the left end strut in an *acute* angle. In general, $R > R'$.

207. Example. Let it be required to find the maximum live-load stresses in a single-track skew bridge from Cooper's E-60 loading, in accordance with the following data:

Pratt truss; span 125 ft.; five 25-ft. panels on center line of bridge; 10-ft. right-forward skew at each end; depth of truss 27 ft.

Stresses. The maximum live-load stresses, in terms of thousands of pounds, are shown in Fig. 9. The stresses in the inclined end posts are based on the shears from Eqs. (4) and (5), and those in the diagonals on the shears from Eqs. (6) and (7). The stresses in the end lower chord members ab and ef are the horizontal components of the stresses in the inclined end posts. Stresses bc and de are found from $\Sigma H = 0$ at b and e for the loading in the same position as for stresses ab and ef, respectively. The other chord stresses are based on Eqs. (10) and (11).

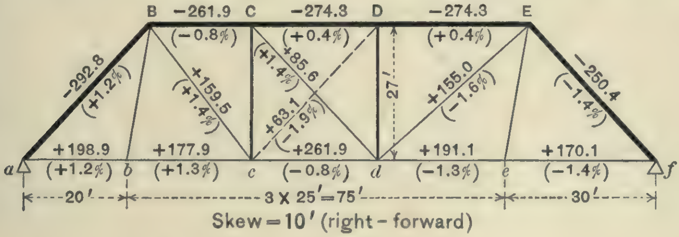


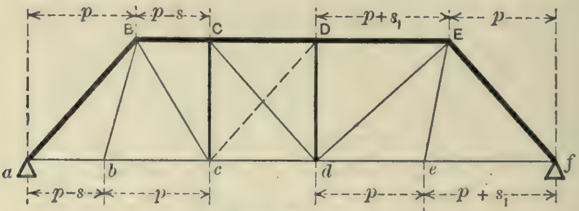
FIG. 9.

The percentages indicate the percentages of the differences between the stresses indicated in the figure and those derived by the approximate formulas, viz., Eqs. (4)a and (5)a for the end posts and end chords, Eqs. (6)a and (7)a for the diagonals, and Eqs. (10)a and (11)a for the intermediate chords.

The stresses in the vertical posts Cc and Dd (not given) are equal to the vertical components of stresses Cd and Dc, respectively. The stresses in the end suspenders *Bb* and *Ee* (not given) may be found as stated in Art. 202, by multiplying the maximum floor-beam reactions at *b* and *e* by the secant of the angle between these members and a vertical.

Since the derivation of these stresses involves no new principles, the computations will not be shown in detail.

208. Middle Chord Stress in Truss with Odd Number of Panels. In a truss with an odd number of panels, as in Fig. 8, Art. 205, the chord stress in the middle panels *cd* and *c′d′* must be found with due regard to which of the diagonals, *Cd* or *Dc*, Fig. 10, is in action. Since the truss is unsymmetrical that diagonal—for example, *Cd*, in panel *cd*—whose lower end is nearer the center of the truss will be in tension from the dead load and the other diagonal, *Dc*, will act as a counter. For maximum stress *CD*(=*DE*), the critical weight must be placed at the moment center *d*, and a position must then be found which will satisfy the criterion, and also cause the live-load shear in panel *cd* to be positive or, if negative, not sufficiently large to neutralize the positive dead-load shear in that panel. If the resultant shear in that panel should be negative, counter *Dc* will act instead of *Cd*, so that the moment center of *CD* will be at *c*, instead of *d*, as assumed (Art. 119).



Truss T

FIG. 10.

In truss *T′* (not shown), *D′c′* is the main diagonal and *C′d′* the counter. For maximum stress *C′D′* (=*B′C′*) in that truss the critical weight must be placed at *c′*, and a position must be sought which will satisfy the criterion and also cause the resultant shear in panel *c′d′* to be negative.

For equal skews the trusses are alike if one truss is assumed turned end for end, i.e., the upper chord members *CD* and *DE* in truss *T* correspond to *C′D′* and *B′C′* in truss *T′*. In that case the stresses in these members should be found for each truss in the manner above described, and the larger stress should be used in proportioning both trusses, since this stress will occur in the other truss when the load is headed in the opposite direction.

209. Special Cases. (*a*) *Skew Equal to Panel Length.* It has been seen (Fig. 3, Art. 201) that when the skew is equal to the panel length, the trusses present the same appearance as for a square bridge. This case is shown in plan in Fig. 11.

Reactions. It is obvious that loads to the left of aa' cannot affect truss T, and that loads to the right of ee' cannot affect truss T'. Therefore, in so far as truss T is concerned, the triangular panel to the left of aa' of which two sides are represented by broken lines, may be assumed omitted and the bridge treated as square ended on the left, and terminating at aa', the reaction $2r'$ taking the place of $2r$. Similarly, with respect to truss T'', the bridge may be regarded as ending at ee' on the right. Therefore in Eq. (4), Art. 205, $s=0$ and in Eq. (5), $s_1=0$. Further, in Eq. (4) s may now be substituted for s_1 and in Eq. (5), l for l'. Introducing these changes in Eqs. (4) and (5),

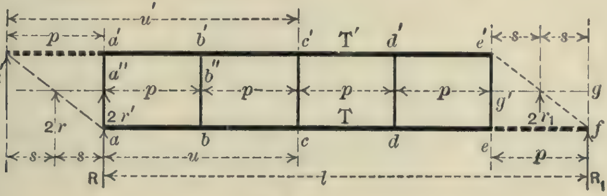


FIG. 11.

$$R=\frac{M_g-r_1s}{l}-r', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$R'=\frac{M_{g'}+rs}{l}-r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In Eq. (12) r' is the reaction at a'' from the loads (on *one* rail) in panel $a''b''$, which may be regarded as the end panel for truss T.

For a bridge of five panels the ratio $s/l=0.1$, and since a truss bridge of fewer than five panels is rarely built, the ratio s/l will in general be less rather than greater than 0.1. Even for this limiting value the terms r_1s/l and rs/l in Eqs. (12) and (13) may be omitted with little error, whence approximately,

$$R = M_g/l - r', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)a$$

$$R' = M_{g'}/l - r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)a$$

For uniform loading on a bridge of five panels the maximum error resulting from the use of these equations is about 1.4 per cent.

The shears may be derived from the reactions and loads in the usual manner.

Bending Moments. Making $s=0$, $s_1=s$, and $r=r'$ in Eq. (10), Art. 207; and $s_1=0$, and $l'=l$ in Eq. (11),

$$M = (M_g - r_1s)u/l - m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$M' = (M_{g'} + rs)u'/l - rs - m. \quad . \quad . \quad . \quad . \quad (15)$$

In Eq. (14) m is now the moment of the loads (on *one* rail) between a'' and the moment center.

These equations may be changed to the simplified forms of Eqs. (10)a and (11)a, Art. 207, without serious inaccuracy.

(*b*) *Deck Girder Bridge with Cross-Ties Resting Directly on the Girders. Reactions.* In this case the panel divisions disappear, and since the cross-ties are laid at right angles to the axis of the bridge, loads to the left of a_1, Fig. 12, will not affect girder G, and loads to the right of g' will not affect girder G'. Therefore, if M_g denotes the moment about g of the loads (on *one* rail) from a_1 to b'', and $M_{g'}$ denotes the moment about g' of the loads (on *one* rail) from a'' to g',

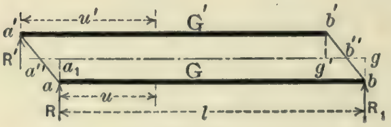


FIG. 12.

$$R = M_g/l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$R' = M_{g'}/l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In these equations it is implied, however, that girder G receives half the load from a_1 to b'', and girder G' half the load from a'' to g', which is far from true with respect to girder G, for the loads to the right of g', and with respect to girder G', for those to the left of a_1. The reactions at a and b' are not materially affected by the inaccuracy of this assumption, but in determining $R'_{\max}$ for the load headed toward the left and in position for the maximum shear at a' (which is equal to $R_{1_{\max}}$ for the load headed toward the right and in position for maximum shear at b) it is advisable to consider the loads to the left of a_1 in their actual positions on the rails and then to ascertain the fractional parts of these loads transmitted to girder G'. In that case $M_{g'}$ in Eq. (17) denotes the sum of the moments about g' of these fractional loads and of the loads (on *one* rail) from a_1 to g'.

Bending Moments. The bending moments may be found with sufficient accuracy by assuming the entire live load applied along the axis $a''b''$. If u and u' denote the distances from a and a', respectively, of the sections at which the bending moments are required,

$$M = M_g u/l - m, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$M' = M_{g'} u'/l - m. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

These equations, which are the same as Eqs. (10)a and (11)a, Art. 207, might have been derived from the general equations, (10) and (11), by making r and r_1 equal zero and substituting l' for l.

It is sufficiently accurate to assume that the maximum bending moments occur at the centers of the girders, and to find the required position of the loading by the criterion of Eq. (28), Art. 42, which becomes for this case, $W = 2W_1$. Letting $u = \frac{1}{2}l$ in Eqs. (18) and (19),

$$M_{\max} = \tfrac{1}{2}M_g - m, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)a$$

and

$$M'_{\max} = \tfrac{1}{2}M_{g'} - m, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (19)a$$

$M_{\max}$ and $M'_{\max}$ may both be found for the load advancing from right to left, and the greater value should be adopted for both girders.

210. Double-track Bridges. The case of a double-track skew bridge with three trusses may be treated in precisely the same manner as that of two separate, single-track bridges, the stresses in the central truss being the sum of the stresses from the loads on each track.

For a double-track skew bridge with two trusses the conditions are different. Referring to Fig. 13, assume the loads applied along the center lines, $t\,t$ and $t't'$, of the two tracks.

Reactions. Considering the loading on track $t\,t$, let $2M_g$ denote the moment about g of the *total* load on the track. The moment about g of the floor-beam loads at b', c', and d' is then $2M_g - 2r(l+s) - 2r_1s_1$. The moment about e of the fractional parts of the loads at b', c', and d' which are transmitted to b, c, and d, may be found by multiplying this moment by v/m. If R_t denotes the reaction at a from the loads on track tt,

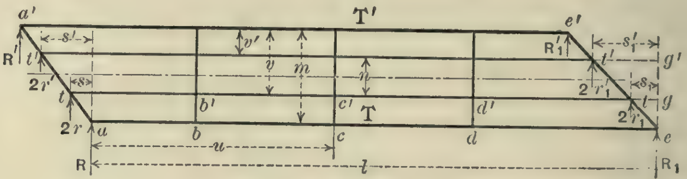


FIG. 13.

$$R_t l - \frac{v}{m}[2M_g - 2r(l+s) - 2r_1s_1] = 0,$$

and since $v = \frac{1}{2}(m+n)$,

$$R_t = \frac{M_g - r(l+s) - r_1s_1}{l} \times \left(1 + \frac{n}{m}\right) \quad . \quad . \quad . \quad . \quad (20)$$

Similarly, if $R_{t'}$ denotes the reaction at a from the loads on track $t't'$, noting that $v' = \frac{1}{2}(m-n)$,

$$R_t = \frac{M_{g'} - r'(l+s') - r_1's_1'}{l} \times \left(1 - \frac{n}{m}\right) \quad . \quad . \quad . \quad (21)$$

For both tracks loaded, $R = R_t + R_{t'}$. The reactions at a' may be determined by means of equations similarly derived.

Eqs. (20) and (21) might have been found, as in Art. 205, by considering the load on each track carried by a pair of stringers a very small distance, Δb, apart, and then taking a longitudinal section of the bridge between each pair of stringers in turn. In this case, however, the shears in the floor beams at such a section are not zero, and these shears must therefore be included in considering $\Sigma M = 0$ about an axis similar to YY in Fig. 7, Art. 205.

Bending Moments. Considering the bending moments at any panel point in truss T, as c, for example, from loads on track $t\,t$, let m_t denote the moment of the total load to the left of c' about that point. Then $m_t - 2r(s+u)$ is the moment about c' of the floor-beam loads to the left of c', of which there exists in this case only a single one at b'. The moment about c of the joint loads in truss T to the left of c is then the product of the above moment by v/m, whence M_t, the bending moment at c from loads on track $t\,t$, has the value

$$M_t = R_t u - \frac{v}{m}\left[m_t - 2r(s+u)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Similarly, $M_{t'}$, the bending moment at c from loads on track $t't'$, has the value

$$M_{t'} = R_{t'} u - \frac{v'}{m}\left[m_{t'} - 2r'(s'+u)\right]. \quad . \quad . \quad . \quad . \quad (23)$$

For both tracks loaded, $M = M_t + M_{t'}$. The bending moment at any panel point of truss T' may be determined by means of equations similarly derived.

B. BRIDGES ON CURVES

211. Clear Width between Trusses. On straight track the 'clear width' of bridges, between trusses, is usually 14 ft. This provides a 'clearance' of 2 ft. on both sides of a car body 10 ft. wide. On curves the width b, between the center lines of trusses, Fig. 14, should be such that the clearances, c_1 and c_2, between the points A and B' of the car, and the members of the truss, shall each have a minimum value of 2 ft. (The width of car body is frequently assumed at 11 ft., and the clearances, for bridges on curves, at 1 ft.) In the figure the car is assumed to be placed at the center of the bridge, as shown in Fig. 15. The point z represents the location of the center of the track at the middle of the bridge, and the point O,

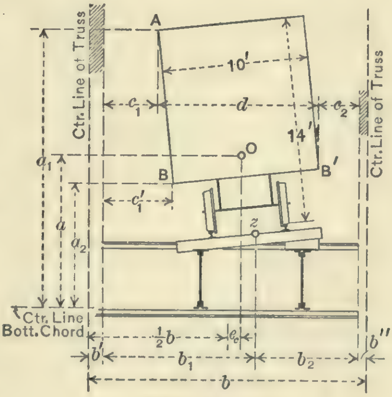


FIG. 14.

the center of gravity of the loaded car. The distance e_c denotes the eccentricity of O with respect to the axis of the bridge. By the A.R.E. specifications the height of the car is assumed to be 14 ft., its length, 80 ft., and the distance between centers of trucks, 60 ft. A car of this size is shown in Figs. 14 and 15. Like points and dimensions in these figures are distinguished by like letters. In Fig. 15 (*b*) XX represents the axis of the bridge, curve (1) the center line of the track, (2) the curve described by the center of gravity of the car, and (3) and (4) the curves described by points B' and A of the car. The distance d between curves (3) and (4) is measured in a direction at right angles to the planes of the trusses, and is assumed to be constant. The distances, e_c and e_e, measured in the same direction between XX and curve (2), represent the eccentricities at the center and ends of the bridge, respectively. In general,

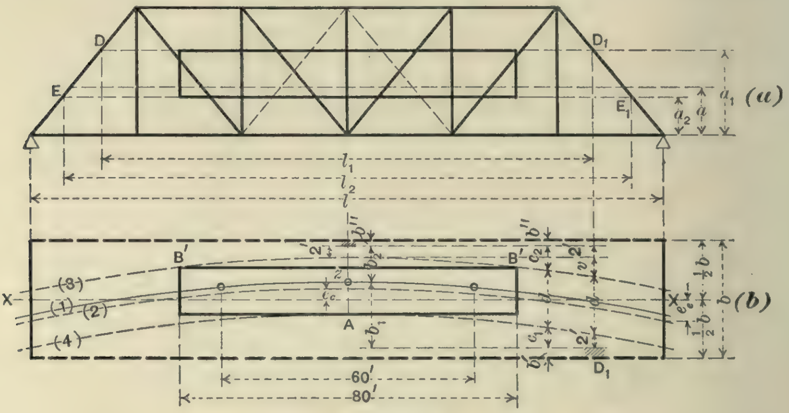


FIG. 15.

$$b = d + c_1 + c_2 + b' + b''.$$

The clearance c_1 is minimum at points D and D_1, Fig. (*a*), a distance l_1 apart, where $c_1 = 2$ ft. The clearance c_2 is minimum at the center of the bridge where $c_2 = 2$ ft. If v denotes the versed sine of curve (3) for the chord length l_1, then, as seen from Fig. (*b*),

$$b = d + v + b' + b'' + 4 \text{ ft.}$$

As the span increases the clearance, c_1', of B, Fig. 14, may become less at points E and E_1, Fig. 15(*a*), than the clearance,

c_1, of A at points D and D_1. In that case curve (4) is the curve described by B instead of A, and v then denotes the versed sine of curve (3) for the chord length l_2. In order to determine the positions of the center lines of the trusses with respect to z, the values b_1 and b_2 must be found, and b' and b'' assumed, the latter being subject to correction after the trusses have been designed. Finally,

$$b = b_1 + b_2 + b' + b''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

Convenient formulas for determining b_1 and b_2 will be derived in Vol. III.

The value of the eccentricity, e_c, at the center of the bridge is then

$$e_c = \tfrac{1}{2}b - (b_2 + b''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

212. Centrifugal Force. According to the A.R.E. Specifications:

"The centrifugal force shall be considered as live load, and it shall be derived from the speed in miles per hour given by the expression $60 - 2\tfrac{1}{2}D$, where D = degree of curve."

The centrifugal force, F_c, of a body moving in a circular path is

$$F_c = \frac{v^2}{gR}W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

in which W is the weight of the body, v the velocity in feet per second, g the acceleration due to gravity (=32.2 ft. per sec., per sec.), and R the radius of curvature in feet. If D denotes the degree of curvature, $R = 50/\sin \tfrac{1}{2}D$, and for $D = 1°$, $R = 5730$. If V is the velocity in miles per hour, $v = \tfrac{5280}{3600}V = \tfrac{22}{15}V$. The centrifugal force, f_c, for a curve of 1° is then

$$f_c = \frac{22^2}{15^2 \times 32.2 \times 5730}V^2W,$$

or

$$f_c = 0.0000117V^2W.$$

Since for comparatively flat arcs R in Eq. (26) may be assumed to vary inversely with D, F_c varies directly as D, whence

$$F_c = Df_c = 0.0000117DV^2W. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

Letting $V = 60 - 2\tfrac{1}{2}D$, the values of F_c for different degrees of curvature and the corresponding velocities are

CENTRIFUGAL FORCE (F_c)

D	V	F_c	D	V	F^c
1	57.5	$0.039W$	6	45	$0.142W$
2	55	$0.071W$	7	42.5	$0.148W$
3	52.5	$0.097W$	8	40	$0.150W$
4	50	$0.117W$	9	37.5	$0.148W$
5	47.5	$0.132W$	10	35	$0.143W$

The resultant centrifugal force, F_c, acts at the level of the center of gravity of the live load, and it may be assumed with sufficient accuracy that its direction is everywhere perpendicular to the planes of the trusses.

213. Vertical Loads on Main Trusses. The vertical loads on the main trusses represent the resultant effect of the vertical live load and the centrifugal force.

(*a*) *Vertical Live Load.* Considering the vertical live load, the fractional part of this load borne by each truss can be found only in an approximate way. For that purpose the load may be assumed applied to the bridge along curve (2) (Fig. 15, Art. 211), which is supposed to be described by the centers of gravity of the units of which the train is made up. The eccentricity of this curve with respect to the axis of the bridge depends on (1) the superelevation of the outer rail, (2) the height of the center of gravity of the loaded car above the rail, and (3) the length of the car between truck centers. Strictly speaking, if the cars are uniformly loaded and in a fixed position, this curve will consist of a series of straight lines representing the horizontal projections of the gravity axes of the loaded cars. The *total* load on each floor beam is evidently not affected appreciably by the curvature of the track, the eccentricity, e, being the only element of uncertainty due to curvature. The eccentricity, e_c, at the center of the bridge, Fig. 16, may be found by Eq. (25), Art. 211, thus locating the reference point z. The

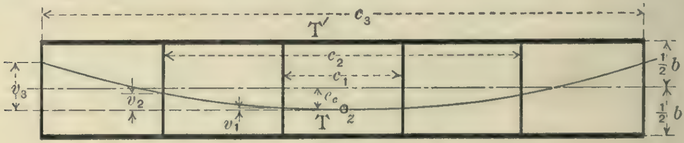


FIG. 16.

versed sines v_1, v_2, etc., at the floor beams, may be found from the corresponding chord lengths c_1, c_2, etc., and the known radius, R, of the curve, by the familiar formula,

$$v = R - \sqrt{R^2 - \tfrac{1}{4}c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

or with sufficient accuracy for practical purposes by the approximate formula,*

$$v = \frac{c^2}{8R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

If D denotes the degree of curvature, $R = 5730/D$. Substituting this value for R in Eq. (29),

$$v = \frac{Dc^2}{8 \times 5730} = \frac{22Dc^2}{1{,}008{,}480},$$

or, closely enough for practical purposes,

$$v = \frac{22Dc^2}{1{,}000{,}000} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)a$$

This convenient formula was derived by H. S. Prichard.

For $c = 150$ ft. and a curve of 8° ($R = 716$ ft.), v by Eq. (28) is 3.939 ft., by Eq. (29), 3.928 ft. and by Eq. (29)a, 3.96 ft.

The corresponding eccentricities e_1, e_2, etc., at successive floor beams from the center are then obtained by combining the values v_1, v_2, etc., from Eq. (29) with e_c.

From the known values of the eccentricities at a given floor beam and at the adjacent floor beams to either side, the mean eccentricity of the loads tributary to the given floor beam may be judged with sufficient accuracy for practical purposes. If W is the total load on a floor beam and e_m its mean eccentricity, the reactions r_v and $r_{v'}$, Fig. 17, at trusses T and T', respectively, are

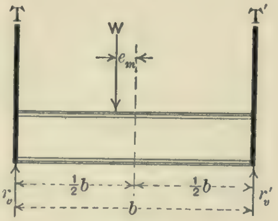


FIG. 17.

$$r_v = \left(\frac{1}{2} + \frac{e_m}{b}\right)W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

$$r_v' = \left(\frac{1}{2} - \frac{e_m}{b}\right)W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

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* For the derivation of this formula, expressed in other symbols, see Art. 275.

By proceeding in this manner with the load on each floor beam, the loads on the trusses may be completely determined.

When the eccentricity lies toward the other side of the center line the reactions may be found by the above equations by giving e a minus sign.

(*b*) *Centrifugal Force.* It was seen in the preceding article that the centrifugal force, F_c, is the product of the vertical load, W, and a numeric factor whose value depends on the degree of curvature and the velocity. Calling that factor q,

$$F_c=qW. \qquad (32)$$

In Fig. 18 let W denote the *total* live load on a floor beam, F.B., and F_c the corresponding centrifugal force given by Eq. (32).

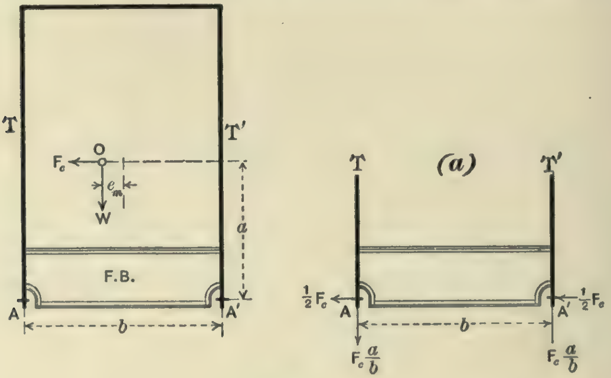

FIG. 18.

This force should be assumed to act at the level of the center of gravity, O, of the live load, i.e., at a height a above the center line of the loaded chord. The effect of F_c acting in the direction shown, is to increase the floor-beam reaction at A from the vertical load by the amount $F_c\,a/b$, and to decrease the reaction at A' by a like amount. Through the development of the couple formed by these two forces, the force F_c may be assumed to be transferred to the plane of the loaded chord (Fig. 3, Art. 15), as shown in Fig. (*a*), where it acts as a horizontal load on the lateral truss. Letting r_c and r_c' denote the reactions from F_c at A and A', respectively,

$$r_c=+qWa/b, \qquad (33)$$

$$r_c'=-qWa/b. \qquad (34)$$

(*c*) *Resultant Loads.* If r and r' denote the resultant loads from W and F_c at A and A' respectively, then from Eqs. (30), (31), (33), and (34),

$$r = r_v + r_c = \tfrac{1}{2}W + \frac{e_m + aq}{b}W, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$r' = r_v' + r_c' = \tfrac{1}{2}W - \frac{e_m + aq}{b}W, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

in which e_m is plus on the side of the center line toward A and minus on the opposite side. The values of q for various degrees of curvature and the corresponding velocities are tabulated in the preceding article. In determining a, the height of F_c above the rail is usually assumed at 5 ft., although it may be much greater.

The joint loads for truss T on the *outer* side of the curve may be found by Eq. (35), but the joint loads for truss T' on the *inner* side of the curve will be greatest when the train speed is so low that the effect of the centrifugal force may be neglected. The joint loads for truss T' for that condition may be derived by Eq. (31), in which e_m may be plus or minus, as pointed out above. Eq. (36) is of interest only in considering the reversal of stress in the lower chord of truss T' under the combined influence of the centrifugal force and maximum wind force in the same direction (Art. 248*b*). In that case, however, W denotes the weight per panel of a train of empty cars.

214. Maximum Live-load Stresses in Main Trusses. In view of the unavoidable uncertainties surrounding other factors, an attempt to find the exact position of the live load for maximum stress in each member is wholly unjustified.

(*a*) *Web Stresses.* The maximum live-load web stresses may be found with sufficient accuracy by transforming the load system into a series of floor-beam loads separated by intervals of one panel length, as explained in Art. 122 (*a*). This new load system may then be applied to the truss with the leading load at the proper joint for maximum shear in each panel in turn. For each position the joint loads on the inner and outer trusses from each floor-beam load may be determined by Eqs. (31) and (35), respectively, and the stresses in those members for which each position governs may then be found by the usual method.

(*b*) *Chord Stresses.* For the chord stresses one of the conventional load systems described in Art. 122 (*b*) may be used. If w is the weight of the equivalent uniform load per foot, per truss, and p the panel length, $W=wp$. The corresponding joint loads for the inner and outer trusses may be obtained by Eqs. (31) and (35) respectively.

215. Vertical Loads on Floor Beams. The maximum live-load bending moments and shears for the floor beams may be found as follows: First find the maximum *total* floor-beam load, W, Fig. 19, as for straight track (Art. 45). Then determine the mean eccentricity, e_m, of the load tributary to the

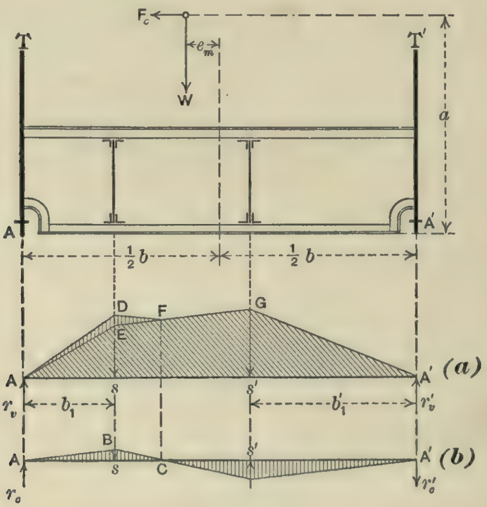

FIG. 19.

floor beam in question as in Art. 213. The reactions r_v and r_v' from W may be found by Eqs. (30) and (31), Art. 213. These reactions are not affected by the positions of the stringers. Then proceed as follows:

Bending Moments. The bending moments M_v and M_v' at stringers s and s', respectively, are

$$M_v=r_v b_1 \quad \text{and} \quad M_v'=r_v'b_1'. \quad . \quad . \quad . \quad (37)$$

In Fig. (*a*), *AEGA'* represents the bending moment diagram for the vertical load W.

The reactions r_c and r_c' from F_c $(=qW)$ may be found by Eqs. (33) and (34), Art. 213, in which a denotes now the height of F_c above the lower flange of the floor beam. The force F_c will develop a downward force at s and an upward force at s', Fig. (*b*). •The bending moment diagram for F_c is shown in Fig. (*b*). The bending moments M_c and M_c' at stringers s and s', respectively are

$$M_c = r_c\,b_1 \quad \text{and} \quad M_c' = r_c' b_1' . \quad . \quad . \quad . \quad . \qquad (38)$$

From A to C the bending moments from F_c are positive, and from C to A' negative. The negative bending moments will tend to reduce the positive bending moments from W, (Fig. *a*). For a slowly moving train these bending moments become so small as to be negligible. The resultant maximum bending moments may be found, therefore, by laying off DE in Fig. (*a*) equal to Bs in Fig. (*b*), and drawing AD and DF. The maximum live-load bending moment at any section of the floor beam is then represented by the ordinate at that section to $ADFGA'$, Fig. (*a*).

Shears. The maximum shears, V and V', for the distances As and $A's'$, respectively, are

$$V = r_v + r_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

and

$$V' = r_v' . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

The effect, $-r_c'$, of F_c on V' is neglected in Eq. (40) on the assumption of a slowly moving train.

216. Vertical Loads on Stringers. The stringers in bridges on curves are eccentric with respect to the axis of the bridge. For flat curves and short spans the alinement of the stringers, ss and $s's'$, Fig. 20 (*a*), is straight, though eccentric. For sharp curves and long spans the stringers are 'offset' at the floor beams, as in Fig. (*b*). The location of the stringers should be determined with respect to the curve described by the center of gravity of the loading, corresponding to curve (2), Fig. 15, Art. 211. The vertical loads on the stringers are due to the weight, W', of the live load in the panel, and the centrifugal force F_c' ($=qW'$).

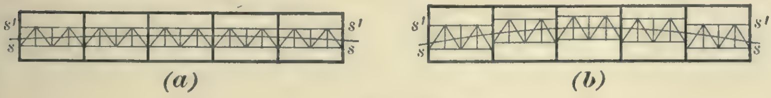

FIG. 20.

The loads from W' must be found with due regard to the eccentricity of curve (2) with respect to the stringer. For that purpose it is sufficiently accurate to consider the curve straight between consecutive floor beams. The effect of F_c' in increasing the load on the outer stringers, $s's'$, may be found by Eq. (33), Art. 213, in which a now denotes the height of F_c' above the top of the stringers, and b, the distance center to center of stringers. The effect of F_c' in decreasing the load on the inner stringers, ss, should be neglected for the reason stated in Art. 213 in connection with the loads on the main trusses.

The position of the live load for maximum shears may be found with sufficient accuracy in the same manner as for straight track, Art. 40 (*c*), and the position for maximum bending moment at the center by Eq. (28), Art. 42, which becomes for this case, $W=2W_1$.

A system of ' stringer bracing,' as shown in Fig. 20, should be provided in the plane of the upper flanges of the stringers, to transmit the horizontal load, F_c', to the floor beams. It is advantageous to arrange this bracing as shown in the figure, because then the stresses in the end braces, which are greatest, are tensile.

217. Stresses in Lateral System from Centrifugal Force. As seen in Art. 213 (*b*), the centrifugal force, F_c, is transferred to the plane of the loaded chord, where it acts as a horizontal load on the lateral truss. Since F_c is a direct function of the vertical live load, W, on *both* trusses (Eq. (32), Art. 213), and since the the panel division of the lateral truss is the same as that of the main trusses, it follows that the loads, F_c, on the lateral truss, as well as the shears and bending moments from these loads, may be found by multiplying the *sum* of each of the corresponding quantities for the *two* main trusses by the factor q, whose value for various degrees of curvature and the corrseponding speeds may be obtained from the table in Art. 212.

CHAPTER XIII

LIVE LOADING FOR BRIDGES

A. RAILROAD BRIDGES

218. Train and Engine Loading. In American practice railroad bridges are generally designed for load systems which consist of a pair of engines coupled and headed in the same direction, followed by a uniform load of indefinite length, representing the train load. The engines are usually of the ‘ consolidation ’ type (a pilot wheel and four drivers on each side) and their weight (including tenders) is represented by a series of concentrated loads. This general form of loading was first adopted in 1875 on the Cincinnati Southern Railroad and soon came into general use. Formerly the engines specified by each railroad were either the heaviest in use on that road, for which the wheel loads and spacings were often prescribed with minute accuracy, or so-called ‘ typical ’ engines were selected in anticipation of future increase. This led to a great variety of loadings, often with unimportant differences, and proved very burdensome to bridge computers. Of late years (since 1896) the load systems proposed by Cooper,* consisting of two typical engines followed by a uniform train load, have come into extensive use, and conditions have thereby been greatly simplified and improved.

219. Cooper's Load Systems have been described in Art. 38. According to the A.R.E. Specifications, the live load shall be “ according to Cooper's series, or a system of loading giving practically equivalent stresses.” By these specifications Cooper's E–40 Loading is the minimum permissible. At present Cooper's E–55 Loading is probably used more extensively than any other by railroads subject to heavy traffic, with a well-defined and increasing tendency toward E–60 Loading.

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* *Train Loadings for Railroad Bridges*, by Theodore Cooper, Trans. Am. Soc. C. E., Vol. XXXI, 1894.

It has been seen (Art. 38), that Cooper's loadings stand in fixed relations to each other. In general:

If w = intensity of uniform train load;

w_1 = mean intensity of engine load for the distance (109 ft.) from the first wheel to the head of the uniform load; and

w_2 = mean intensity of load on drivers for the distance (5 ft.) between centers of drivers; then

$$w : w_1 : w_2 :: 1 : 1.3 : 2.$$

220. Special Alternative Loadings. Provision should be made for occasional extremely heavy loads on single car trucks which may produce excessively high stresses in short floor girders and secondary web members. The A.R.E. Specifications prescribe for that purpose a pair of axle concentrations, 7 ft. between centers, each 25 per cent heavier than the heaviest axle concentration in the engine.

Again, in the design of turntables it is customary to assume an especially elongated wheel base, thereby increasing the stresses when the engine is in position for turning with its center of gravity vertically above the center of the table.

221. Increase in Live Loads. The increase in the weight of engines and freight cars has been so rapid that it has usually been found necessary to strengthen or to replace railroad bridges within 15 to 25 years after their construction.

In 1875, the maximum loading was that specified by the Cincinnati Southern Railroad, consisting of two 69-ton engines followed by a train load of 1820 lbs. per foot. This is seen to be less than one-third as great as Cooper's E–60 Loading, which consists of two 213-ton engines followed by a train load of 6000 lbs. per foot.

Locomotive Loads. In Fig. 1, line (1) represents the heaviest specified axle loads for the years 1886 to 1903, as given by Hodge,* based on typical rather than actual engines. The heaviest axle loads cited by Hodge for locomotives in actual service are those for certain 195.7–ton consolidation engines on the Pittsburgh, Bessemer & Lake Erie R.R., on which the axle loads run from 55,000 lbs. to a maximum on one of the axles of 59,200 lbs.

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* *Live Loads for Railroad Bridges*, by Henry W. Hodge, Trans. Am. Soc. C. E., Vol. LIV., Part A, 1905.

Line (2) represents the heaviest axle loads for engines built by the Baldwin Locomotive Works from 1885 to 1907, as given by Wille.* The upper limit of the latter, 53,500 lbs., has been since exceeded in a consolidation engine built at these works for the Erie Railroad, for which the axle loads range from 54,200 lbs. to a maximum on one of the axles of 65,400 lbs. Again, in a locomotive of the Atlantic type (two pairs of drivers), built recently by the Pennsylvania R.R., the axle loads reached the unprecedented weight of 68,770 lbs.

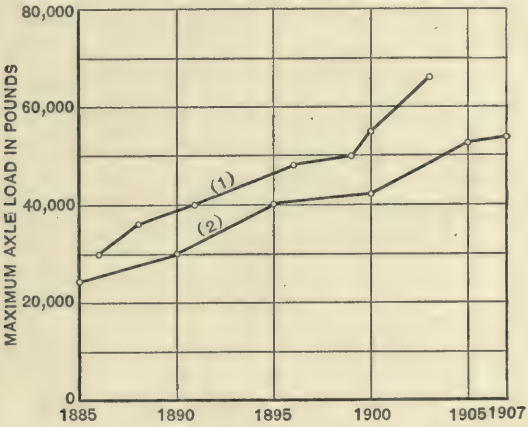

FIG. 1.

Line (1) represents an increase from 30,000 to 66,000 lbs., i.e., 120 per cent, for a 17-year period, and line (2) an increase from 24,000 to 53,000 lbs., i.e., 123 per cent, for a 22-year period.

TABLE I

AVERAGE WEIGHT OF CONSOLIDATION ENGINES

(EXCLUSIVE OF TENDERS)

Year.	Total Weight. Lbs.	Total Weight on Drivers. Lbs.	Weight on One Driving Axle. Lbs.
1885	112,000	97,000	24,250
1907	200,000	179,000	44,750
Increase in 22 years	79%	85%	85%

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* Discussion on *Steel Rails*, by H. V. Wille, Proc. Am. Soc. Test. Mats., Vol. VII, 1907.

The increase in the *average* weight of consolidation engines *exclusive of tenders*, from 1885 to 1907, on the basis of all engines of that type built by the Baldwin Locomotive Works for regular railroad service in the United States, as given by Wille, is as shown in Table I.

Train Loads. The increase in the weight of train loads during the past twenty-five years has been no less remarkable, the specified uniform train load having increased during that period from 3000 to 6000 lbs. per foot, i.e., 100 per cent. The loaded weight of the modern steel coal car ranges from about 4000 to 4500 lbs. per foot, and ore cars are in use whose loaded weight is about 7300 lbs. per foot.

Articulated Locomotives. There appears to be good ground for the belief held by many engineers that Cooper's E–60 Loading is not likely to be greatly exceeded for many years. The demand for high-power locomotives has been met within recent years by the construction of engines of the Mallet articulated type with six, eight, or ten pairs of drivers instead of four as in the consolidation type, thus securing greatly increased tractive power without increasing the load per axle. The axle concentrations for the largest locomotive of that type—and at present (1911) the largest locomotive in the world of any type—is shown in Fig. 2. Several of these engines are in service on the Atcheson, Topeka & Santa Fé Ry. Considering the relative length of this locomotive and two Cooper E–60 engines, the comparison is as shown in Table II.

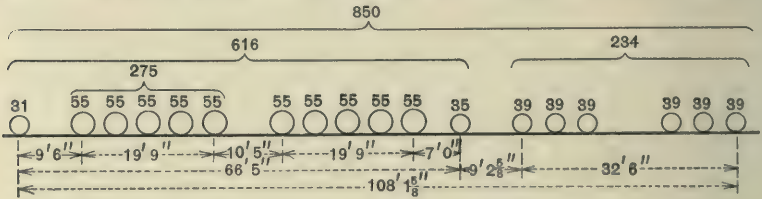

FIG. 2.—425-ton Mallet Articulated Locomotive.

TABLE II

Type.	Total Weight. Tons.	Total Length. Feet.	Average Weight. Lbs. per Feet.
A.T. & S.F.	425	108.1	7860
2 Cooper E-60	426	104	8190

Although the average weight per foot, as well as the maximum load per driving axle, is greater for the E-60 Loading, the grouping of the ten 55,000-lb. driving-axle concentrations in the A.,T. & S.F. loading is such that for a certain range of short spans it will produce greater stresses.

222. Conventional Equivalent Load Systems. Various forms of loading have been proposed to replace load systems of the general type considered above. A few of these systems will be briefly noticed.

(*a*) *Uniform Load with a Single Excess Load.* For maximum bending moments the uniform load is assumed to extend over the whole bridge and the single 'excess' load is placed at the vertical through the moment center. If w denotes the intensity of the uniform load and W the single excess load, the effect is evidently the same as from a uniform load of intensity $w+2W/l$ covering the entire bridge (Art. 42*b*).

For maximum web stresses the uniform load is treated by the conventional method (Art. 98), and the excess load is added at the loaded joint farthest from the support at the loaded end. For girders without panel divisions the excess load is applied at the head of the uniform load.

This form of loading was proposed by Pegram in 1886,* and is used to-day on the Pennsylvania Lines West of Pittsburgh. The uniform load is 5000 lbs. per foot, and the concentrated load 60,000 lbs.

(*b*) *Uniform Load with Two Excess Loads.* In this form of loading two equal excess loads about 50 ft. apart—i.e., separated by a distance roughly equal to the distance between the centers of gravity of two consolidation locomotives—are used.

For maximum bending moments the uniform load and one excess load are disposed as in (*a*), and the second excess load is placed at that joint in the longer segment of the span whose distance from the moment center is nearest to 50 ft. For maximum web stresses the uniform load and one excess load are treated as in (*a*), and the second excess load is placed at that joint in the loaded segment of the span whose distance from the first excess load is nearest to 50 ft.

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* Trans. Am. Soc. C. E., Vol. XV., 1886, p. 474, and Vol. XXI, 1889, p. 575; also *Engineering News*, Vol. XXV, No. 2, 1891.

(*c*) *Uniform Load with Intermediate Series of Concentrated Loads.* This form of loading has been used on the Norfolk and Western Ry. since 1891,* and is now used on that road in the form shown in Fig. 3.

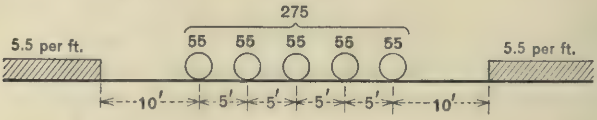

FIG. 3.

(*d*) *Equivalent Uniform Loads.* This form of loading will be treated in the next article.

223. Equivalent Uniform Loads. Given a load system consisting of a pair of engines followed by a uniform load, an *equivalent uniform load* may be found which, when applied to the bridge in the position for maximum value of a given function, will produce the same maximum value as the given load system. The mean intensity of engine loading for the entire distance covered by the engines is always greater than the intensity of the uniform train load, and for short distances, occupied only by the drivers, it is much greater. The relative intensities for Cooper's loadings are stated in Art. 219. Evidently the relative effect of the excess weight of the engines as compared with the train diminishes as the distance covered by the loading increases. The equivalent uniform load therefore decreases as the span increases, or rather as the distance covered by the live load in position for maximum value of a given function increases. The equivalent uniform load is, however, also somewhat affected by the distribution of the concentrated loads over a given distance. Thus for maximum chord stresses in general the loading must be distributed over the entire bridge. The equivalent uniform load for maximum chord stress differs, however, somewhat for successive panels, its tendency being to decrease from the ends of the bridge toward the center.

(*a*) *Beams.* Consider, for example, the maximum bending moments and shears at various sections of a 100-ft. girder without panel divisions, Fig. 4 (*a*).

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* Trans. Am. Soc. C. E., Vol. XXXI 1894, p. 189, and Vol. LIV, 1904, p. 98.

The points on the dotted curve, Fig. (*a*), are found by laying off the maximum bending moments from Cooper's E–60 Loading as ordinates at the one-eighth points. The maximum bending moment at the center is represented by M. The curve ACB is a parabola with vertex at C, and it is therefore the bending moment diagram for an equivalent uniform load which will give the same bending moment at the central section as the E–60 Loading. At all other sections the true curve of maximum bending moments is seen to lie somewhat outside this parabola.

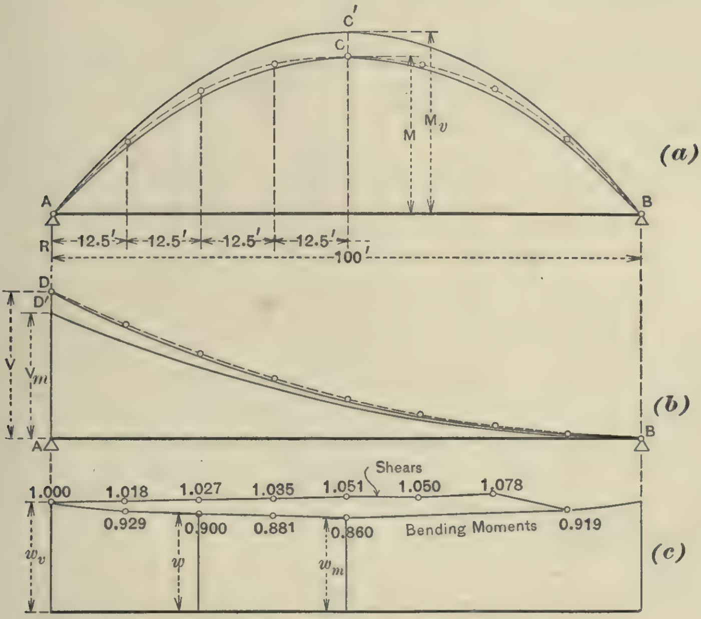


Fig. 4.

Similarly, in Fig. (*b*) the dotted curve is the true curve of maximum shears from E–60 Loading. The parabola BD is the curve of maximum shears from an equivalent uniform load which will give the same maximum shear, V, at the end section. As shown in Art. 33, V is also the slope of the tangent at A to the dotted curve ACB in Fig. (*a*). The curve $AC'B$ in that figure is a parabola whose tangent at A has the same slope, V, i.e., it is the moment diagram of an equivalent uniform load which will produce the same maximum end shear V as the E–60 Loading.

The curve BD', Fig. (*b*), is a parabola for maximum shears from the equivalent uniform load which produces the bending moment M at the center. If w_m and w_v denote the intensities of the equivalent uniform loads corresponding to M and V, respectively, $M_v/M = V/V_m = w_v/w_m$. For E–60 Loading and a 100-ft. span, $w_v/w_m = 1.16$.

The ordinates to the curves in Fig. (*c*) denote the intensities of the equivalent uniform loads for maximum positive shears and maximum bending moments at the one-eighth points of a 100-ft. beam for E–60 Loading. The numbers indicate the ratios of these intensities to w_v. The line for shears is not continued beyond the seven-eighths point. The ordinate to this line immediately to the left of B would be infinity.

Fig. (*c*) shows that for bending moments w_v leads to errors on the side of safety, and w_m to errors on the side of danger. A compromise is therefore sometimes made by computing the bending moments on the basis of an equivalent uniform load, w, which will give the same maximum bending moments at the quarter points as the actual loading. To find w, let M denote the maximum bending moment at the quarter points from the concentrated load system, and l the span length. Then $M = (\frac{1}{2}wl \times \frac{1}{4}l) - (\frac{1}{4}wl \times \frac{1}{8}l) = \frac{3}{32}wl^2$, or

$$w = \frac{32\,M}{3\,l^2}. \qquad \qquad (1)$$

With respect to shears, Fig. (*c*) shows that w_m leads to errors on the side of danger, which become relatively larger for successive panels from the end. It is to be observed, however, that the positive shears in the right half of a beam from a load advancing toward the left are counter shears which have no practical significance.

Although the above analysis is limited to a 100-ft. girder and E–60 Loading, the curves in Fig. (*c*) are perfectly characteristic, and the conclusions stated are generally applicable to beams and, in somewhat modified form, to trusses, for live loads of the same general character.

(*b*) *Trusses.* With regard to maximum bending moments at the joints, the conditions in a truss are as a rule identical with those in a beam, and in the exceptional cases very nearly so. If the intensity, w, of the equivalent uniform load is so chosen that it will give the same maximum shear, V, in the end panel as the given load system, it will also produce the same stress in the end chord member. To meet the requirement just stated, the relations between V and w are given by the equation

$$V=\tfrac{1}{2}w(l-p),$$

in which l is the span length and p the length of the end panel in question, whence,

$$w=\frac{2V}{l-p}. \qquad (2)$$

For a truss of equal panels, let $l=Np$, whence,

$$w=\frac{2V}{(N-1)p}. \qquad (3)$$

The stresses from this equivalent loading in all chord members except those in the end panels will involve errors on the side of safety. For Cooper's loadings, and for trusses from 125 to 500 ft., in length, these errors usually run from 2 to 6 per cent. If the equivalent uniform load is found by Eq. (1), that is, if it is based on the maximum bending moment at the quarter-point section, the range of error is from about ± 1 to ± 3 per cent.

With regard to maximum web stresses the conditions of loading in a truss approximate those for maximum shear in a beam. If the equivalent uniform load for web stresses is so chosen that it will give the same maximum shear, V, in the end panel as the given load system, the web stresses in all other panels will be subject to small errors on the side of danger if these stresses are computed by the exact method for uniform loading, as given in Art. 124, for trusses with horizontal chords, and in Art. 138, for trusses with inclined chords. If, however, the stresses are determined by the conventional method for uniform loading (Arts. 98 and 134), the neglect of the half-panel load at the forward joint will cause the net errors to be on the side of safety. While as a rule the percentage of error increases rapidly in successive panels from the end, the absolute errors are inconsiderable, the conditions being somewhat as illustrated for a truss with horizontal chords in the example in Art. 124, and for a truss with inclined chords in Art. 139. (See also Arts. 126 and 127.)

224. Stresses by Equivalent Uniform Loads. The stresses in ordinary railroad bridges are seldom determined on the basis of equivalent uniform loading, except as an approximate check for the chord stresses computed from concentrated load systems (Art. 116). If this method is used, however, it seems best that the equivalent load which produces the same maximum shear in the end panel as the given load system should be adopted for all principal truss members. If the web stresses are then found by the conventional method all differences will be on the side of safety. These differences are comparatively small, and since the live-load stresses based on any load system whatever will differ to a considerable and uncertain extent from the maximum stresses developed in actual service, there is really no valid ground for supposing that the stresses derived from an equivalent uniform load in the manner above suggested will accord less closely with the stresses from the actual loading than the stresses obtained by more refined assumptions as to the distribution of the live load.

If a closer agreement is desired between the stresses from (1) a given train and engine load, and (2) equivalent uniform loads, a series of values for (2) may be derived which will give, respectively, precisely the same bending moments as (1) at the various even fractional division points (corresponding to panel points in trusses with equal panels) for any desired range of span length. Similarly, a series of values for (2) may be derived which will give precisely the same shears (by the conventional method) as (1). By plotting these equivalent uniform loads for shears and bending moments for a series of span lengths with a suitable range of fractional panel divisions, the equivalent loads for intermediate span lengths may be read from the diagram with sufficient accuracy. The web stresses in trusses with inclined chords would, however, still be subject to very slight inaccuracies, if based on the equivalent uniform loads for maximum shears in corresponding panels.

Instead of plotting the equivalent uniform loads, the bending moments and shears corresponding to these loads may be plotted directly in the manner explained. The range of these values is, however, so much greater than that of the intensities of the uniform loads, that for diagrams of approximately the same size the readings are subject to much greater percentages of error.

The substitution of an equivalent uniform load for a given train and engine load system is especially advantageous in the analysis of stresses under somewhat complex conditions, as for square or skew bridges supporting tracks on curves, arches, swing bridges, etc. The use of equivalent loads in such cases is not only advisable on account of the great simplification of the computations, but it is justifiable because of the increased uncertainty to which the stresses are then inherently subject.

225. General Comparison of Train and Engine Load Systems with Other Load Systems. The relation between the computed and actual live-load stress in a bridge member for a given load system depends largely on the dynamic effect of the live load. The dynamic effect is influenced by many conditions, such as speed, irregularity of track, flat wheels, lack of balance of the reciprocating and rotating parts of the locomotive mechanism, vertical component of stress in connecting rod, and cumulative vibrations from different causes. Moreover, the train and engine loadings are usually typical rather than actual ones, and the latter are subject to frequent change. It is evident therefore that great refinement in the computation of live-load stresses is really unwarranted. Many forms of equivalent loads have been proposed as substitutes for train and engine loads with a view of simplifying the computation of stresses. No single one of these substitute systems has been widely adopted, however. From the standpoint of the railroads the train and engine-load system provides in a simple, safe and easily understood manner not only for all ordinary conditions, but also for the many special conditions frequently encountered in practice. If the train and engine-load system deserves to be regarded as the most suitable standard of comparison for other systems, as seems to be generally conceded, it is doubtful whether any simple form of equivalent loading can be derived which will lead to substantially the same results under all circumstances.

As has been seen, the use of train and engine loads may be greatly simplified by various labor-saving devices. It is desirable also that specifications for bridges should contain tabulated data for shears and bending moments in beams, floor-beam reactions, etc., for spans within the usual limits.

The labor and time required for computing the live-load stresses from train and engine loading is often greatly overstated. After a little practice, and with the aid of a moment table and slide rule, these stresses may be found for a five or six-panel truss with horizontal chords in less than one hour, and for a seven-panel truss with inclined chords in about two hours.

226. Loads for Double-track and Multi-track Bridges. The improbability of simultaneous maximum loading on more than one track increases both with the number of tracks and the length of the span. By the A. R. E. Specifications the percentage added to the computed live-load stresses for impact (Art. 102) is therefore decreased as the loaded length of track for maximum stress in a given member increases. It is specified further that "for bridges carrying more than one track, the aggregate loaded length of all tracks producing the stress shall be used."

Cooper's Specifications provide:

> "For any greater train loading than E–40, all double-track bridges with two trusses, for spans 150 ft. and over, may be proportioned for the higher train loading on one track, and E–40 loading on the second track."

The allowance to be made for live loading in the case of long-span bridges, carrying say four or more tracks for railroad or electric car service in addition to roadways and sidewalks, may involve not only the consideration of suitable intensities for simultaneous loads of different kinds, but also (for spans of extraordinary length) the lengths of the different load units, and whether more than one continuous load unit shall be assumed to act at the same time on different parts of the same track with the intervening distance free from loading, for members in which the stress is thereby increased. Such matters must be judged according to the circumstances of each case, and cannot be fixed by general rules.

B. HIGHWAY BRIDGES

227. Classification of Highway Bridges. With respect to live loading other than the weight of pedestrians, highway bridges may be classified in a general way as follows:

I. City bridges subject to heavy vehicular traffic and
  (*a*) Trains of heavy electric motor passenger and freight cars, or single motor cars followed by trailers, or
  (*b*) Heavy electric cars in single-car units.

II. Suburban and interurban bridges subject to less heavy vehicular traffic than Class I, and (*a*) heavy, or (*b*) light electric cars in single-car units.

III. Country bridges subject to (*a*) moderately heavy, or (*b*) light vehicular traffic, in either case with or without light electric cars in single-car units.

228. Live Loads for Floors. The flooring and floor joists receive their maximum loads under the passage of a single electric car, traction engine, road roller or motor truck. The transverse floor beams receive their maximum loads when the two panels on either side of the floor beam are fully loaded.

In Cooper's Specifications for Highway and Electric Street Railway Bridges,* such bridges are classified as follows:

Class A, City Bridges.

Class B, Suburban or Interurban Bridges,

Class C (no descriptive designation).

Class D, Country Highway Bridges.

Class E_1, Electric Street Railway Bridges with Heavy Equipment.

Class E_2, Electric Street Railway Bridges with Light Equipment.

The live loads for "the floor and its supports" according to Cooper's Specifications are shown below in Table III.

Let W = total load in tons on two axles, 10 ft. between centers,

and w = intensity of uniformly distributed load, in pounds per square foot of floor surface.

The load W is assumed to occupy a space 30 ft. long, with a width of 12 ft. for single-track and 22 ft. for double-track electric car lines. The loads in Cooper's Specifications according to the above classification may then be tabulated as follows:

---

* New and revised edition, by Bernt Berger (1909).

TABLE III

Class.	*W.*		*w.*	
	On Any Part of Roadway } or {	On Each Street-car Track.	On Remaining Portion of the Floor, Including Foot-walks.	On Total Floor Surface.
A	24	24	100	..
B	12	24	100	..
C	12	18	100	..
D	6	..	...	80
E_1	..	24	...	..
E_2	..	18	...	..

Note. Loads *W* and *w* are to be combined in every case except for Class *D*.

The live loads for the floor are applicable also to such secondary truss members as receive their maximum live-load stress when a single floor beam receives its maximum load.

229. Live Loads for Trusses. In the case of highway bridges the intensity of the live load governing the stresses in the main members of trusses may consistently be reduced as the span length increases, owing to the increasing improbability that the maximum intensity of loading for short lengths will be realized as the length increases. The live loads prescribed for trusses in Cooper's Specifications for bridges of the several classes referred to in the last article, and for different span lengths, are shown in Fig. 5, in which

w_1 = pounds per lineal foot of each car track,

w_2 = pounds per square foot of remaining floor surface.

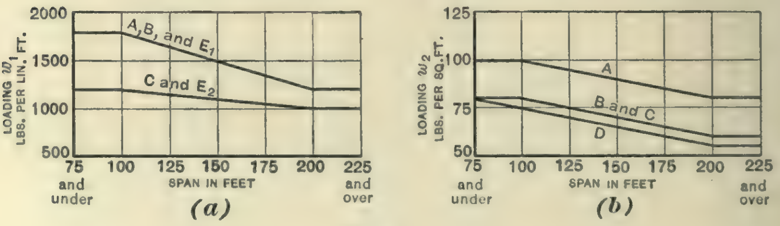


FIG. 5.

In estimating the area of the "remaining floor surface," the occupied width for single track is to be taken at 12 ft. and for double track at 22 ft., as stated in the last article. The maximum live-load stress in each member is to be found either from w_1 or w_2 according to which of these loads gives the greater value when placed in the position for maximum stress in the member in question. As seen in the last article the concentrated loading on the track or roadway is to be *combined* with the uniform loading on the remaining floor surface in determining the maximum loads on individual floor beams, and in those secondary truss members in which the stress becomes maximum when a single floor beam receives its maximum load. On the other hand, for the main truss members the condition of concurrent maximum loading on the tracks, roadway and sidewalks is disregarded by reason of the extreme improbability that such loading will ever occur over any considerable length of the bridge.

It will be noted that the maximum load per foot of track in Fig. (*a*) is 1800 lbs., and that this is somewhat greater than the average weight per foot of the heaviest load W (48,000 lbs.) in the last article, for the distance (30 ft.) which this load is supposed to occupy; for $48{,}000/30 = 1600$.

230. Maximum Loads. The maximum loads for highway bridges in or near large cities have been increasing rapidly within recent years. The heaviest electric motor cars weigh, when loaded, about 100,000 lbs. and have an over-all length of about 50 ft., the average load per foot of track being therefore about 2000 lbs.

The wheel spacing and axle loads of the 53-ton electric motor cars on the Long Island R.R. is shown in Fig. 6. The heaviest axle loads, 30,800 lbs., are seen to be as great as the maximum axle loads of consolidation locomotives in 1890 (Fig. 1, Art. 221).

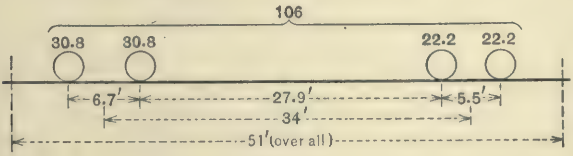


Fig. 6.

The electric motor cars of the Interborough Rapid Transit R.R. weigh, when loaded, about 51.3 tons, and those of the Brooklyn Rapid Transit R.R., about 46.8 tons, the over-all lengths and wheel spacings differing only slightly from those in Fig. 6.

When the live load consists of trains of electric motor cars, or of single motor cars followed by trailers, the same loads per foot of track should be used for all spans within the limit of the maximum train length. If the axle loads at one end are much heavier than at the other, as in Fig. 6, the bending moments become maximum when one car is assumed to be turned end for end, and the four heavy axle loads are then placed in the vicinity of the moment center. This form of loading will also govern for the shears at and near the ends of girders, and in the end panels of trusses.

There is now an increasing tendency to haul heavy coal cars and miscellaneous freight cars over electric railroads. Such loading may greatly exceed the weight of electric motor cars. Bridges for such traffic are frequently designed for Cooper's E–35 or E–40 Loading.

CHAPTER XIV

LATERAL BRACING AND TRANSVERSE BRACING

231. Definitions. The bracing between girders and trusses in the planes of one or both chords is called *lateral bracing.* The bracing in vertical or inclined planes between girders or between corresponding web members in trusses is called *transverse bracing.* The transverse bracing at the ends of through bridges is called 'portal bracing,' and that at the ends of deck bridges, 'end bracing.' The transverse bracing at intermediate points is called 'sway bracing.'

232. Object of Bracing. In general, bracing is provided *to maintain the girders or trusses in their true relative positions during erection and in service.* The bracing should be so proportioned and so disposed as to enable the structure to resist the action of the external forces to which it is subject, as nearly as possible as an *integral whole.* Usually this may be best accomplished by making the individual members of the bracing rigid.

Specifically, bracing is provided for the following purposes:

1. To transmit wind loads, centrifugal forces (for bridges on curves) and tractive forces to the supports.
2. To resist lateral forces developed by the lurching movement of rapidly moving locomotives, due chiefly to the unbalanced action of certain parts of the mechanism.
3. To minimize vibrations from rapidly moving loads in general.
4. To support the compression chords laterally, thereby reducing their slenderness ratios as columns and increasing the allowable working stresses.

The effect of the lurching movement of locomotives may be said to be the governing condition for the bracing in short-span railroad bridges. While the stresses from this cause cannot be determined in an exact quantitative sense, specifications provide in a more or less arbitrary manner for this element, and for vibrations caused by rapidly moving loads in general, by prescribing much higher loads for the lateral systems than can be developed by the wind pressure alone (Art. 234).

233. Wind Pressures. The subject of wind pressure has been treated in Arts. 66 to 68 inclusive, in connection with roofs and buildings, which present relatively much greater surfaces to the wind than bridges. In considering the allowances for wind pressure on bridges it is especially important to note (*a*) that the simultaneous intensities of wind pressure on various parts of a large surface may differ widely, and (*b*) that the maximum average intensity of pressure decreases as the area of the surface increases.

On account of (*a*) some specifications treat the wind pressures on the bridge itself, as well as that on the live load, as a moving load, on the extreme assumption that the maximum wind pressure may be developed on certain parts of a structure while the simultaneous pressure on the remainder of the structure is zero. On account of (*b*) the intensity of the specified wind pressure may consistently be reduced as the span increases.

234. Specified Lateral Forces. (*a*) *Railroad Bridges.* According the A. R. E. Specifications:

> "All bridges shall be designed for a lateral force on the loaded chord of 200 lbs. per linear foot plus 10 per cent of the specified train load on one track, and 200 lbs. per linear foot on the unloaded chord; these forces being considered as moving."

By Cooper's Specifications:

> "To provide for the wind and vibrations from high-speed trains:
>
> "The top lateral bracing in deck bridges, and the bottom lateral bracing in through bridges shall be proportioned to resist a lateral force of 600 lbs. for each foot of the span; 450 lbs. of this to be treated as a moving load, and as acting on a train of cars, at a line 6 ft. above base of rail.
>
> "The bottom lateral bracing in deck bridges, and the top lateral bracing in through bridges, shall be proportioned to resist a lateral force of 200 lbs. for each lineal foot of spans up to 200 ft., and 25 lbs. additional for each additional 50 ft."

The comparison between the above requirements may be stated as follows:

Loaded Chord. By the A. R. E. Specifications the lateral force is made a function of the train load for reasons mentioned in Art. 232. Thus for Cooper's E-60 Loading, the lateral force is $200+(6000\times0.10)=800$ lbs. per foot of span, all of which is to be treated as a moving load.

By Cooper's Specifications, the lateral force, irrespective of the specified live load, is 600 lbs. per foot of span, of which only 450 lbs. is to be treated as a moving load.

Unloaded Chord. The A. R. E. Specifications prescribe a lateral force of 200 lbs. per foot of span, regardless of the span length or the area of the surface exposed to the wind. This load is to be treated as a moving load.

Cooper's Specifications provide for a constant lateral force of 200 lbs. per foot of span, for spans up to 200 ft., and progressive increases for longer spans. Thus for spans of 250, 500, 750, and 1000 ft., the lateral forces per foot of span are 225, 350, 475, and 600 lbs., respectively. These loads are to be treated as fixed loads.

(*b*) *Highway Bridges.* For highway bridges up to a span of 300 ft., Cooper's Specifications provide for a lateral force of 300 lbs. per foot of span for the loaded chord, of which 150 lbs. is to be treated as a moving load; and 150 lbs. per foot of span for the unloaded chord. For spans above 300 ft., the lateral force for each chord is to be increased by 10 lbs. per foot of span for each additional 30 ft.

235. Types of Lateral Systems. The lateral systems of plate-girder bridges are usually of the triangular type (Fig. 29 (*b*), Art. 107), the diagonals being designed to develop the maximum stresses both in compression and tension, to provide for lateral forces acting on either side of the bridge. The triangular system is also frequently used for short riveted trusses, especially for deck bridges, for which the width center to center of trusses is less than for through bridges. In general, however, the lateral systems of truss bridges, including the longest spans, are of the Pratt type, Fig. 1. The diagonals are usually rigid members, consisting of single or double angles, or of four angles connected in pairs by latticing, according to the magnitude of the stress and the length of the member. Their slenderness ratio is frequently so great that their strength in compression is nominal and therefore neglected. For the lateral forces acting in a given direction, only that one of the pair of diagonals in a given panel in which the stress is tensile is assumed to be in action. Thus for the symmetrical lateral loading shown in Fig. 1, the diagonals represented by full lines are in tension. If the diagonals are sufficiently rigid to act effectively in compression, the shear in each panel from the lateral forces is assumed to be borne in equal parts by the two diagonals in that panel, although the stresses are then, strictly speaking, statically indeterminate.

The floor beams serve as struts in the lateral system in the plane of the loaded chord. The design of the struts in the upper lateral system of through bridges is governed chiefly by the consideration that these struts as well as their connections to the top chord must be sufficiently strong and rigid to resist (with the assistance of the diagonals) the tendency of this chord to bend laterally as a whole rather than as a single-panel compression member.

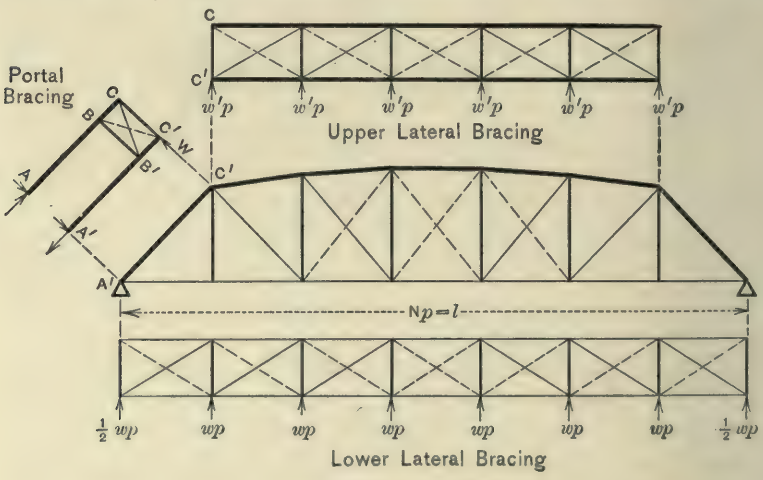


FIG. 1.

The top chords of pony trusses must for like reasons be supported laterally at every joint by some form of knee-brace connected to the floor beams.

236. Assumed Distribution of Lateral Forces. Refined assumptions as to the distribution of the lateral forces are unwarranted. Thus, it is sufficiently accurate to assume that the lateral forces are applied wholly to the windward truss, and that the load at the hip joint, C', Fig. 1, is equal to the loads at the intermediate upper chord joints. If w' denotes the intensity of the lateral load on the top chord, and W denotes the sum of the joint load $w'p$ at C' and the loading transmitted to that joint through the upper lateral system,

$$W=\tfrac{1}{2}(N-1)w'p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The load W must be conveyed to the supports A and A' by the portal bracing and end posts. In reality this load acts partly at B, B', and C, but the assumption that it is applied wholly at C' involves only slight errors on the side of safety, and is sufficiently accurate for practical purposes.

237. Stresses in Lateral Bracing. The lateral bracing together with the chords of the main trusses may be regarded as independent lateral trusses supporting the lateral forces. If the main vertical trusses lie in vertical planes, the chords of the lateral trusses are parallel. If one or both chords of the main trusses are inclined, the chord stresses from the lateral forces will induce stresses in the web members of the main trusses. In general, however, the stresses in the main trusses due to the lateral forces are relatively so small in comparison with the stresses from the dead and live loads that they may be neglected. The possible reversal of stress in the lower chord, from tension to compression, under the influence of the lateral forces, must frequently be investigated, however, as will be seen later (Art. 248).

The lateral loading is always regarded as a uniform load. If the loading is treated in part as a fixed load covering the whole bridge, as in Cooper's Specifications, the stresses may be found as in Art. 83 for vertical loads. If the loading is treated wholly or in part as a moving load, the web stresses may be found in the usual way from the maximum shears by Eq. (57), Art. 124, as in the example in that article, the chord stresses being the same as for a fixed load. If the unloaded chord is inclined, the shears may be computed as for a truss with horizontal chords, and the stresses in the diagonals of the lateral truss may then be found by multiplying the shear in a given panel by a/b, in which a is the true length of the diagonal and b the width of the bridge, center to center of trusses.

The stresses in lateral bracing from centrifugal forces, in bridges on curves, have been treated in Art. 217.

238. Portal: Load and Reactions. As previously stated, it is sufficiently accurate to consider the load on the portal to consist of a single horizontal force W, Fig. 2, applied at its upper windward corner, C'. If the truss consists of N equal panels of length p, and w' denotes the intensity of the assumed wind pressure on the upper chord, then from Eq. (1), Art. 236,

$$W = \tfrac{1}{2}(N-1)w'p.$$

The force W and an equal force at the other extremity of the upper chord produce upward vertical reactions, R, at the supports on the leeward side, equal to $W \times h/b$, and downward vertical reactions, R', on the windward side, Fig. (b), also equal to $W \times h/b$. Resolving these reactions into components in the directions of the inclined end posts and the lower chord, as in Fig (a), $R_v' = R'a/h = Wh/b \times a/h = Wa/b$. The component reactions, R_v and R_v', in the plane of the portal are shown in Fig. (c).

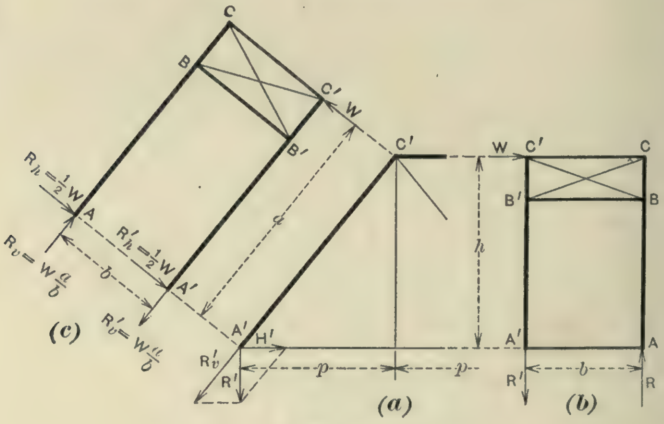
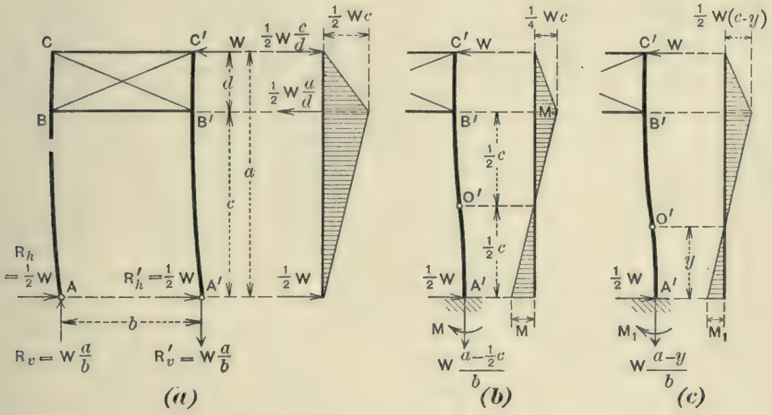


FIG. 2.

If the lower chord of the truss slopes downward from the support, let h denote, as in Fig. 2, the vertical height of C' above A', and h' the vertical distance from C' to the lower chord. The vertical reactions are then Wh/b, as before, but now R_v and $R_v' = Wh/b \times a/h' = Wa/b \cdot h/h'$.

The transverse horizontal reactions, R_h and R_h' at A and A', respectively, Fig. (b), are statically indeterminate, but the usual assumption that R_h and $R_h' = \frac{1}{2}W$ is sufficiently accurate. The external forces acting upon the portal consist, then, of the load W and the four reactions R_h, R_h', R_v, and R_v'.

239. Effect of End Conditions at Bottom of Posts. In the analysis in the preceding article it is implied that the end posts are hinged at the bottom. The force W will then cause the portal to deflect as in Fig. 3(*a*). The horizontal forces, $\frac{1}{2}W$, at A and A' develop horizontal reactions, $\frac{1}{2}Wc/d$, at C and C', and $\frac{1}{2}Wa/d$ at B and B'. These forces produce bending moments in the posts, as shown in the shaded moment diagram. The bending moment is evidently maximum at B and B', where its value is $\frac{1}{2}Wc$.



(*a*) (*b*) (*c*)

FIG. 3.

If the conditions are such that the posts may be regarded as fixed-ended at A and A', the portal will deflect as in Fig. (*b*). If the portal frame is so rigid that its distortion may be neglected, then, as explained in Art. (75)*b*, the bending moments at B and B' are one-half as great as in Fig. (*a*), and equal (but opposite in character) to the bending moments M at A and A', i.e., $M = \frac{1}{4}Wc$. Further, the bending moments at O and O', at a distance $\frac{1}{2}c$ from the bases, are zero. The vertical reactions, $W(a - \frac{1}{2}c)/b$, at A and A' are now the same as for a portal of a height $a - \frac{1}{2}c$ (measured in the plane of the portal) and hinged at the bases. Since the horizontal reactions are unaffected by the height, the stresses may be analyzed precisely as in Fig. (*a*) after substituting $\frac{1}{2}c$ for c and $a-\frac{1}{2}c$ for a.

If an end floor beam is riveted between the posts, the latter may be regarded as fixed-ended at A and A'. If no end floor beams are provided, the posts may be regarded as fixed-ended if the dead-load stress, S_d, in each post is sufficient to develop a resisting moment of $\frac{1}{4}Wc$. Assume, for example, that each post has two bearings on the end pin, a distance k apart, between centers, as in Fig. 4. Then, if $\frac{1}{2}S_dk > \frac{1}{4}Wc$, the posts may be considered fixed-ended. Suppose, however, that $\frac{1}{2}S_dk(=M_1) < \frac{1}{4}Wc$. Then, as shown in Fig. 3 (*c*), $y < \frac{1}{2}c$, where y denotes the distance from the bases to sections O and O' at which the bending moment is zero. In that case $y = M_1/\frac{1}{2}W$, i.e.,

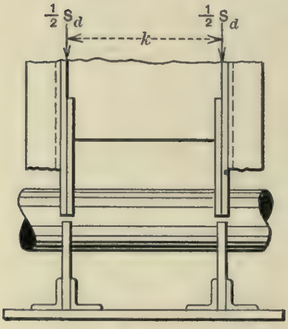


FIG. 4.

$$y = \frac{S_dk}{W}. \qquad (2)$$

The stresses may be analyzed precisely as in Fig. (*a*) after substituting $c-y$ for c and $a-y$ for a. The vertical reactions are now the same as for a portal of a height $a-y$, hinged at the bases.

A condition of complete or partial fixed-endedness at the bases not only leads to a reduction of stress in the portal bracing, but the consequent diminution of the vertical reactions has an important bearing on the question of reversed stresses in the lower chord, as will be seen later (Art. 248).

In the following articles relating to stresses in portals the posts will always be treated as hinged at the bases. The resulting formulas may then be applied to other end conditions by assigning suitable values to a and c, as just explained.

For reasons to be considered later (Art. 248) the wind stresses in the end posts do not, under ordinary conditions, affect the sectional areas required for these members. With rare exceptions these posts are made of the same cross-section throughout, so that if the stresses from wind are considered, the combined effect of the axial and bending stresses need be determined only at sections B and B', where the bending moment is maximum. Its value at these sections is always the product of $\frac{1}{2}W$ by the distance from B and B' to the section at which the bending moment is zero. The bending moments in the end posts will therefore not be further considered in subsequent analyses of portal bracing.

240. Portal with Diagonal Bracing. This form of portal, Fig. 5, is especially well adapted to large bridges, the diagonals being made rigid, with riveted connections at the ends and at their intersection.

The load W, Fig. 5, is held in equilibrium by the reactions at A and A'. The horizontal reactions will develop horizontal forces F_1 at C and C', and F_2 at B and B'. With respect to the bracing these forces act in the directions shown in the figure, but with respect to the end posts they act in the opposite direction, as previously shown (Fig. 3 a) in connection with the moment diagram for the posts. The conditions are precisely the same as explained in Art. 75 for roofs supported on columns and stayed laterally by knee-braces.

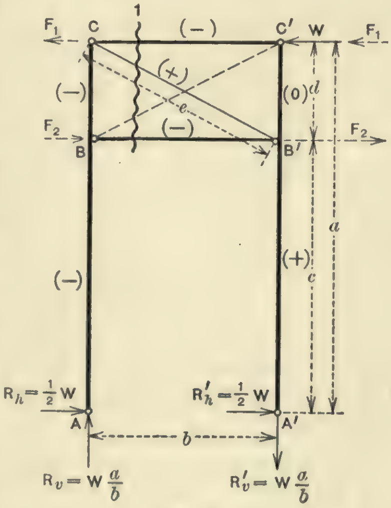


FIG. 5.

Evidently

$$F_1 = \frac{1}{2}W\frac{c}{d} \quad \text{and} \quad F_2 = \frac{1}{2}W\frac{a}{d} = F_1 + \frac{1}{2}W, \quad . \quad . \quad (3)$$

and the posts are subject to the bending moments shown in Fig. 3 (a). The axial stresses may be found as follows:

(a) *Algebraic Method.* Let it be assumed that the diagonals can act only in tension. Then for W in the direction shown

BC' may be regarded as removed. From $\Sigma V=0$ at A, B, A', and C', the stresses in the posts are

$$AB=BC=-W\frac{a}{b}. \qquad A'B'=+W\frac{a}{b}, \quad . \quad . \quad . \quad (4)$$

$$B'C'=0.$$

From $\Sigma H=0$ at B and C', respectively,

$$BB'=-F_2=-\frac{1}{2}W\frac{a}{d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$CC'=-(F_1+W)=-\left(F_2+\frac{1}{2}W\right)=-\frac{1}{2}W\left(\frac{a}{d}+1\right). \quad . \quad (6)$$

From $\Sigma V=0$ at B',

$$CB'=+W\frac{a}{b}\cdot\frac{e}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Stresses BB' and CC' may also be found by taking section 1, and considering the moments of the forces R_h and R_v to the left of the section about the appropriate moment centers C and B', respectively.

(*b*) *Graphic Method.* Referring to Fig. 6, bisect CC' at D, and draw AD. Lay off $\overline{DE}=\frac{1}{2}W$ to any convenient scale, and drop the vertical EF. Draw FG horizontally to its intersection G with DG drawn parallel to CB'. Drop the perpendicular DH, and bisect GH at I. The stresses may then be scaled as follows:

$$AB=BC=A'B'=\overline{EF} \qquad B'C'=0$$

$$CB'=\overline{DG}, \qquad BB'=\overline{HI}, \qquad CC'=\overline{FI}.$$

Proof. The load W and the resultant reactions R and R' are in equilibrium. These three forces must therefore intersect at a common point (Art. 16). The reactions R and R' have equal horizontal components and equal vertical components. Hence R and R' must have the same slope, and must therefore intersect W at D. By construction $\overline{EF}$ is the vertical component of R and R', and therefore the stress in AB, BC, and $A'B'$. From

$\Sigma V=0$ at C', stress $B'C'$ equals zero. Triangle DGH is the force triangle of the forces at C, $\overline{DH}$ representing stress BC; hence stress $CB'=\overline{DG}$. From Fig. 5, stress $BB'=-F_2$; therefore from $\Sigma H=0$ at B', the horizontal component of stress $CB'=2F_2$. Hence in Fig. 6, stress $BB'=\overline{HI}$. From Eqs. (5) and (6), stress $CC'=$stress $BB'+\frac{1}{2}W$; therefore stress $CC'=\overline{FI}$.

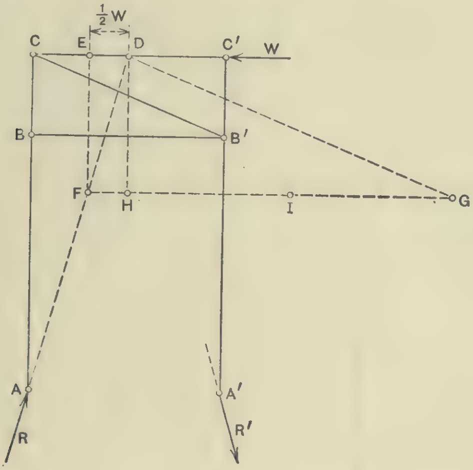


FIG. 6.

241. Portal with Triangular Bracing. This type, in its most general form, is shown in Fig. 7. Other forms to be considered afterwards are more efficient, however, and are used much more frequently.

Axial Stresses. The stresses will first be completely stated. If the stresses in symmetrically disposed members differ only in sign, the sign ± will be used, the particular sign for each stress being shown in the figure. As before, $F_1=\frac{1}{2}Wc/d$, and $F_2=\frac{1}{2}Wa/d=F_1+\frac{1}{2}W$. The stresses are as follows:

$$AB=A'B'=\pm W\frac{a}{b}. \qquad (8)$$

$$CD=+F_1=+\frac{1}{2}W\frac{c}{d}. \qquad (9)$$

$$C'D'=-(F_1+W)=-W\left(\frac{c}{2d}+1\right). \qquad (10)$$

$$BD = B'D' = \pm F_2\frac{e}{f} = \pm\frac{1}{2}W\frac{a}{d}\cdot\frac{e}{f}. \qquad (11)$$

$$BC = B'C' = \pm P_1\frac{g}{b} = \pm F_2\frac{d}{f}\cdot\frac{g}{b} = \pm\frac{1}{2}W\frac{a}{b}\frac{g}{f}. \qquad (12)$$

$$DD' = -\frac{1}{2}W. \qquad (13)$$

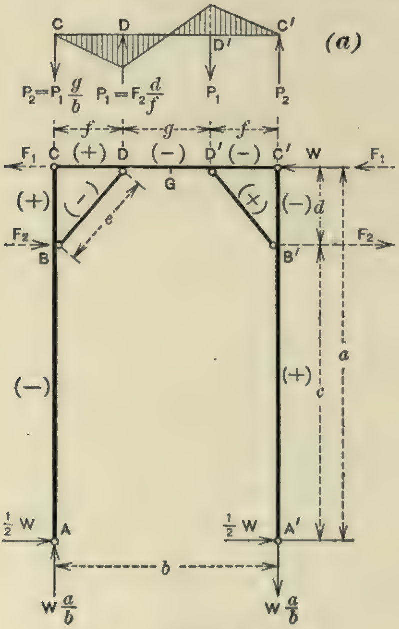


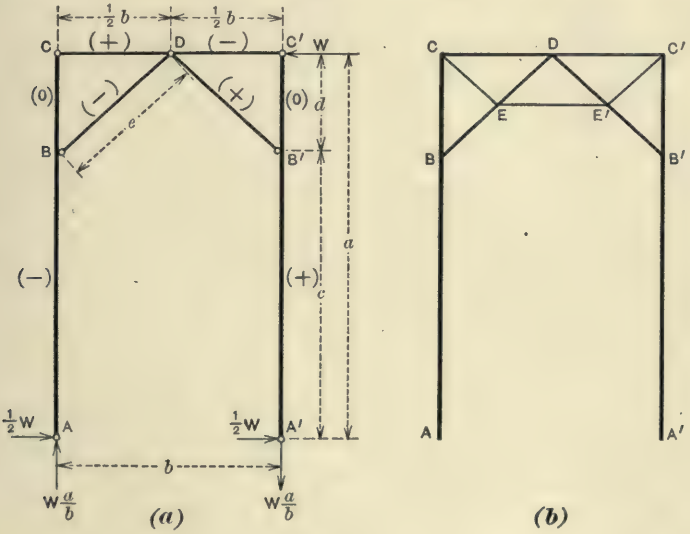
FIG. 7.

With the explanation in the preceding article, the derivation of these stresses is for the most part obvious. Stress $C'D'$, Eq. (10), equals stress CC', Eq. (6), although it is expressed in a different form consistent with Eq. (9), rather than with Eq. (5). From $\Sigma H = 0$ at B and B', the horizontal component of stresses BD and $B'D'$ is F_2. Hence the vertical component P_1 of these stresses is $F_2 d/f$. Since DB is in compression and $D'B'$ in tension the forces P_1 constitute a couple which acts on the beam CC' as shown in Fig. (a). The reactions P_2 at C and C' are therefore $P_1 g/b$. With respect to the posts the forces P_2 act in directions opposite to those indicated in the figure, stress BC being therefore tensile and stress $B'C'$ compressive. From $\Sigma V = 0$ applied at C and C', stresses BC and $B'C'$ would appear to be zero. The frame is, however, not a true truss. If joints D and D' were hinged, the frame would collapse. Stability is maintained partly by the resistance of CC' to bending, and it is this bending action which gives rise to stresses BC and $B'C'$. Finally, stress DD' may be found by taking a section at G, midway between D and D', where the bending moment is zero, and considering the forces to the left of the section. The force $\frac{1}{2}W$

must be balanced by a compressive stress $\frac{1}{2}W$ in DD', and the vertical reaction Wa/b produces a vertical shear of like magnitude in that member.

Bending Stresses. The moment diagram for CC' is shown in Fig. (a). The maximum bending moment occurs at D and D', and its value is $P_2 \times f = F_2\, dg/b = \frac{1}{2}Wag/b$. The bending moments in the posts are as shown in Fig. 3 (a), Art. 239.

A more efficient form of portal is that shown in Fig. 8 (a). Since the knee-braces BD and $B'D$ meet at a common point, the top strut CC' is not subject to bending. While that strut is always built as a continuous member, the portal would be stable if joint D were hinged.



(a) (b)

FIG. 8.

Stresses AB, $A'B'$, CD, and $C'D'$ may be found by Eqs. (8), (9), and (10). From $\Sigma V = 0$ at C and C', stresses BC and $B'C$ are zero. This also follows by letting $g = 0$ in Eq. (12). From $\Sigma V = 0$ at B and B',

$$BD = B'D = W\frac{a}{b}\cdot\frac{e}{d}, \quad \ldots\ldots\ldots \quad (14)$$

Eq. (14) may be derived from Eq. (11) by letting $f = \frac{1}{2}b$.

The form of portal shown in Fig. (*b*) is very efficient, and is frequently used for bridges of moderate length. The primary members are the same as in Fig. (*a*). The secondary members CE, EE', and $E'C'$ are added for the sake of greater rigidity and better appearance. These members are made relatively light, and their presence is assumed to have no effect on the stresses in the primary members. Strictly speaking, this is true only if the horizontal reactions at A and A' are equal, as assumed. For then the lines of action of the resultant reactions at the bases traverse D, as in Fig. 6, and since D is the moment center of stress EE', that stress is zero. Consequently stresses CE and $C'E'$ are also zero. Horizontal members, from E and E' to the end posts, are frequently added, although this practice is of doubtful value. Such members cannot contribute materially to the rigidity of the portal, and they affect the stresses in the other members to a degree which is statically indeterminate.

A portal of this type compared with that in the preceding article offers the important advantage that the bottom connection points B and B' may be brought considerably below the lowest admissible level of the clearance line EE' at the middle, thus increasing the rigidity of the portal and reducing the bending moments in the end posts.

242. Plate-girder Portal. This form of portal, Fig. 9 (*a*), consists of a plate girder, riveted to the trusses in the vicinity of the hip joints. In the following analysis it will be assumed that the shear is resisted wholly by the web plate, and that the bending moments are resisted wholly by the flanges. The distance d is the 'effective depth' of the girder, i.e., its depth between centers of gravity of the flanges.

The stresses and bending moments in the posts AB and $A'B'$ are the same as in the previous cases. The forces acting on the girder are completely shown in Fig. (*b*). The forces F_1 and F_2 are the components of the horizontal reactions R_h and R_h'. The vertical reactions are transmitted directly to B and B'.

Stresses BB' and CC' may be completely determined by combining the direct stresses from the horizontal forces F_1 and F_2 to the left of any section with the stresses S, due to the bending moment, M, of the vertical force, R_v, to the left of the section. At a section 1, distant x from the left end, $M = Wa/b \times x$ and $S = M/d = Wa/b \times x/d$. For $x = b$, $S = Wa/d$. The stresses S

vary uniformly from zero at the left end to Wa/d at the right end, as shown graphically in Fig. (*b*). Stress BB' is evidently $-F_2$ at the left end, and $+F_2$ at the right end, and stress CC' is $+F_1$ at the left and $-W+F_1$ at the right end. Since the stresses S vary uniformly with x, whereas the forces F_1 and F_2 to the left are constant, the resultant flange stresses must vary uniformly between the above values at the end sections, as shown graphically in Fig. (*b*). As a check, note that the arithmetic sum of the end ordinates equals Wa/d for each flange.

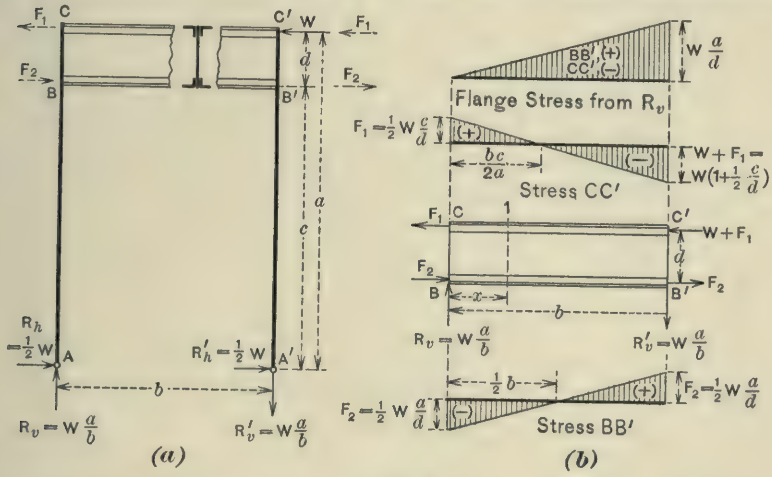


FIG. 9.

The flange stress is maximum at C', and its value there is

$$\text{Maximum flange stress (at } C') = -W\left(1+\frac{1}{2}\,\frac{c}{d}\right). \quad . \quad (15)$$

The force R_v produces a shear Wa/b at every vertical section of the girder.

Portals of this type are usually reinforced at the corners B and B' by brackets between the girders and the posts. While such brackets add materially to the strength and rigidity of the portal, their influence on the stresses is usually disregarded, especially since it is statically indeterminate. If the brackets are substantially proportioned, the section of maximum bending moment in the end posts may be assumed to lie at the lower end of the bracket. The fact that this type of portal, as well as the lattice type, to be considered in the next article,—unlike the diagonally braced portal (Art. 240),— admit of the use of efficient brackets or knee-braces which tend to reduce the bending moments in the end posts, is an important advantage in their favor.

243. Lattice Portal. A common form of lattice portal is shown in Fig. 10. For reasons stated in the last article, the effect of the brackets, shown by broken lines, on the stresses in the portal will be disregarded, although their influence on the bending moment in the end post may properly be taken into account. The diagonals must be sufficiently rigid to act effectively in compression. The vertical shear, $R_v(=Wa/b)$, which is constant throughout the portal, will be assumed to be borne equally by the two diagonals traversed by any vertical section. Thus,

$$(\text{Stresses in diagonals}) = \frac{1}{2}W\frac{a}{b}\cdot\frac{e}{d} = (\pm)W\frac{ae}{2bd}. \quad . \quad . \quad (16)$$

The stress in any given diagonal is tensile or compressive according to whether W acts at C' or, in the opposite direction, at C.

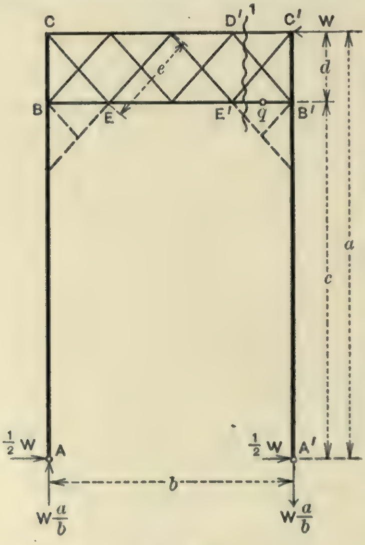


FIG. 10.

The chord stresses vary, in a general way, like the flange stresses in the plate-girder portal, Fig. 9 (*b*). With W acting at C' the chord stress is maximum in $C'D'$, and since this member is in compression, for which the allowable unit stress is lower than for tension, the sectional area of the upper chord will be determined by the requirements for that member. In the lower chord the maximum compressive stress, which occurs in member BE, is somewhat lower than that in $C'D'$, but it is sufficiently accurate to make both flanges of the same size, as is customary in practice. It is only necessary, therefore, to find stress $C'D'$.

Taking section 1, assume the moment center at q, midway between B' and E'. Since stresses $B'D'$ and $C'E'$ are equal and opposite in character, and since their lever arms with respect to q are equal, their resultant moment about that point is zero. From moments about q of the forces to the left of the section,

$$D'C' = -\left(W\frac{a}{b}\cdot\frac{7}{8}b - \frac{1}{2}Wc\right) \div d,$$

or

$$D'C' = -W\left(\frac{7}{8} + \frac{3}{8}\,\frac{c}{d}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

The same result might have been found from $\Sigma H = 0$ at C'. The vertical component of the compressive stress in $C'E'$ is $\frac{1}{2}Wa/b$. Therefore its horizontal component is $\frac{1}{2}Wa/b \times \frac{1}{4}b\ d = \frac{1}{8}Wa/d$. Subtracting this from the sum of W and the horizontal force $\frac{1}{2}Wc/d$, developed at C' by the reaction $\frac{1}{2}W$ at A',

$$D'C' = -\left(W + \frac{1}{2}W\frac{c}{d} - \frac{1}{8}W\frac{a}{d}\right),$$

or, letting $a = d + c$,

$$D'C' = -W\left(\frac{7}{8} + \frac{3}{8}\,\frac{c}{d}\right),$$

as before.

Comparing Eqs. (17) and (15) it is seen that the maximum chord stress is somewhat less than the maximum flange stress in a plate-girder portal. The difference decreases as the number of panels in the lattice portal increases. Since the difference is on the side of safety, and since the analysis is at best only an approximate one, Eq. (15) may be used in general for the chord stresses in portals with numerous diagonals, as in Fig. 11.

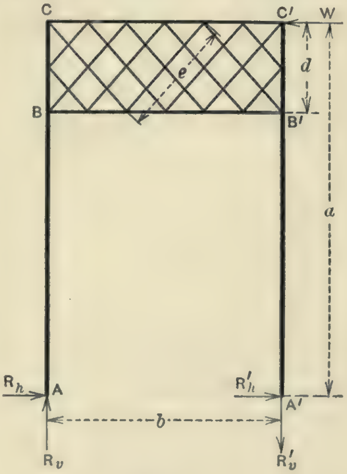


FIG. 11.

The stresses in the diagonals may be found by Eq. (16) after substituting N, the number of diagonals cut by a vertical section through the portal, for 2 whence in general,

$$(\text{Stresses in diagonals}) = \pm W\frac{ae}{Nbd}. \quad . \quad . \quad . \quad .(18)$$

Portals of this form are usually provided with knee-braces, not shown in the figure.

244. Skew Portal. The portal of a left-forward skew bridge is shown in plan in Fig. 12 (*b*). The end posts have equal slopes and their horizontal projections AC and $A'C'$ are equal to p, the panel length along the axis of the bridge (Fig. 2, Art. 201). The lateral force W, normal to the plane of the trusses at C', must be resolved into two components, W_1 and W_2. The component W_1 in the direction of the portal is Wb_1/b. Fig. (*c*) represents the portal projected on a plane parallel to itself, i.e.,

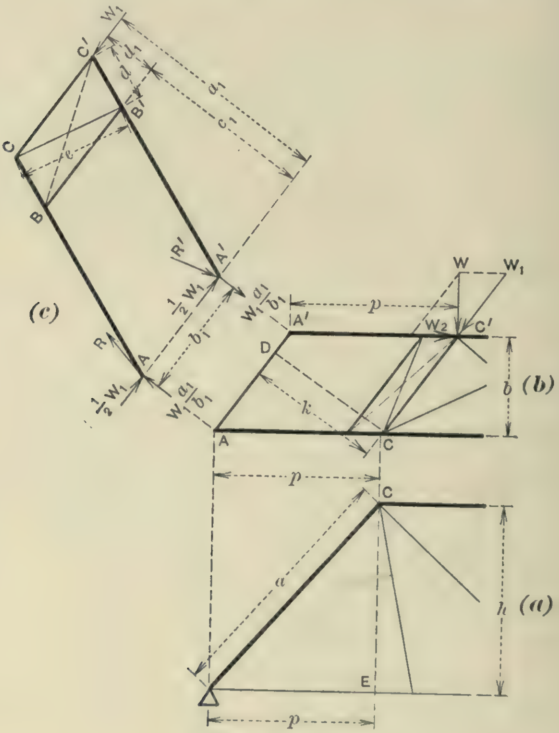


FIG. 12.

revolved into the plane of the paper about axis AA'. The perpendicular distance a_1 is the hypotenuse of a right triangle of which $\overline{CD}(=k)$, drawn from C at right angles to AA', Fig. (b), is the base, and $CE(=h)$, Fig. (a), is the altitude. Thus,

$$a_1=\sqrt{k^2+h^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which $k=pb/b_1$.

The reactions at A and A' have the values shown in Fig. (c). The resultant reactions, R and R', may be resolved into components in the direction of the end posts and at right angles to that direction. The former are the direct stresses in AB and $A'B'$, while the latter produce bending in these members. The stresses BB' and CC' may be found by Eqs. (5) and (6), Art. 240, after substituting a_1 for a, d_1 for d, and $\frac{1}{2}W_1+F$ for $\frac{1}{2}W$, in which F is the horizontal component of the vertical reaction, W_1a_1/b_1 at A', the other component lying in the direction $A'B'$. Finally, stress $CB'=$stress $A'B'\times e/d$.

245. End Bracing in Deck Bridges. The lateral forces on the loaded upper chord of deck bridges are much larger than those on the unloaded upper chord of through bridges (Art. 234). For a deck bridge with tension diagonals in the end panels, Fig. 13, if w' denotes the intensity of the lateral load on the upper chord,

$$W=\tfrac{1}{2}Nw'p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

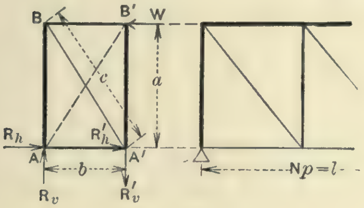


FIG. 13.

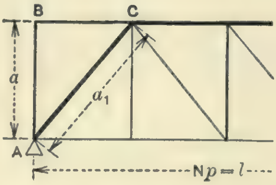


FIG. 14.

This force must be transmitted to the supports by the bracing between the vertical end posts. If AB' cannot act efficiently in compression, the stresses are as follows:

$$BB'=-W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$A'B=+W\frac{c}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

If the conditions at A' are such that horizontal reaction, $R_h' = W$, can be developed at that support, stress AA' is zero. The general assumption that $R_h = R_h' = \frac{1}{2}W$ is in this case inadmissible. In practice usually no attempt is made to compute the stress in AA', but a substantial strut is provided to keep the bases in their true relative positions.

In the design of deck bridges of moderate span it frequently happens that the slenderness ratio of the end diagonals is such that they can act efficiently in compression. In that case the horizontal shear W may be assumed to be borne equally by both diagonals. Then $R_h = R_h' = \frac{1}{2}W$ and stress AA' is zero, as before.

If the diagonals in the end panels are compression members, as in Fig. 14, it is best to make provision for transmitting the lateral loads on the top chord—except the force $\frac{1}{2}w'p$ at B—to the supports through bracing between the inclined end posts AC. In this case

$$W = \tfrac{1}{2}(N-1)w'p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

as for a through bridge, and if c_1 denotes the length of the diagonal in the plane AC, its stress is $+Wc_1/b_1$. The stresses in the bracing in the plane AB are found by letting $W = \frac{1}{2}w'p$ in Eqs. (21) and (22).

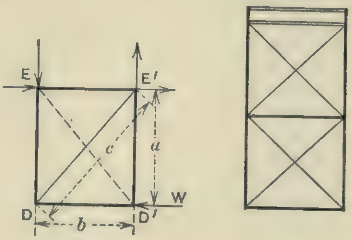
Strictly speaking, the lateral forces can be transmitted in the manner above indicated only if the upper laterals in the end panel BC are omitted. For proper rigidity under rapidly moving loads it is important, however, that adequate bracing should be provided in that panel. In that case the lateral forces on the top chord instead of being transmitted wholly to C and thence, by the bracing in the plane AC, to A, will be carried partly to B and thence to A by the bracing in the plane AB. The proportions in which the load will be transmitted along these two paths are statically indeterminate. The inclined posts AC are, however, much heavier members than the vertical posts AB, and if the bracing in the plane of the former is also made relatively heavy in comparison with that in the plane of the latter, the proportion of the lateral load that will be transmitted from C by the indirect path, CB–BA, is relatively small. It is advisable that this element should be disregarded entirely in proportioning the bracing in the plane AC, and that the stresses in the diagonals in that plane should be based on the value of W by Eq. (23).

For reasons above stated, it is also desirable, however, in proportioning the bracing in the plane AB, that the lateral force $\frac{1}{2}w'p$ at B should be increased in a somewhat arbitrary manner, according to judgment and the circumstances in each case, to a value between the limits, say, of $0.15w'l$ and $0.25w'l$. The stress in the diagonals is then found by substituting that value for W in Eq. (22). It is to be observed in that connection that specifications always prescribe minimum permissible sizes for bridge members, and that these sizes are in general greater than required for a stress in the diagonals based on a lateral force of only $\frac{1}{2}w'p$ at B.

246. Sway Bracing for Single-track Bridges. If complete systems of upper and lower lateral bracing are provided, as well as portal or end bracing, the lateral forces can all be transmitted to the supports through these systems. To ensure stability, sway bracing at intermediate points is then not needed. For practical reasons, however,—that is, since sway bracing conduces greatly to the general rigidity of a bridge,—it should always be used for deck bridges, and so far as the overhead clearance requirements permit, for through bridges.

(*a*) *Deck Bridges.* For deck bridges of moderate span the sway bracing consists usually of a single pair of diagonals, Fig. 15 (*a*), in the planes of the principal compression members of the main trusses or, in the case of plate-girder bridges, at intervals of from one to three panel lengths of the lateral system.

For short deck plate-girder bridges, the diagonals in the lower lateral bracing are frequently omitted. The lateral forces on that chord are then transmitted to the upper lateral bracing by the sway bracing. If p denotes the distance between consecutive planes of sway bracing, and w the intensity of the lateral loading on the lower chord, then, in Fig. 15 (*a*), stress $DD' = -wp$, and stress $E'D = +wpc/b$. Stress EE' depends on



(*a*) (*b*)

FIG. 15.

the horizontal reactions at E and E' whose sum is wp. If the upper lateral system is of the triangular type, the entire reaction wp will be developed at either E or E', and that is true also at that panel point of a Pratt truss at which the shear in the lateral system passes through zero. In the usual case of deck bridges with transverse floor beams, the latter also act as struts for the lateral and sway bracing. The diagonals of the sway bracing occupy the position relative to the floor beams shown in Fig. (*b*), and therefore produce compression in the lower flange, thus tending to reduce the tension in that flange from vertical loading.

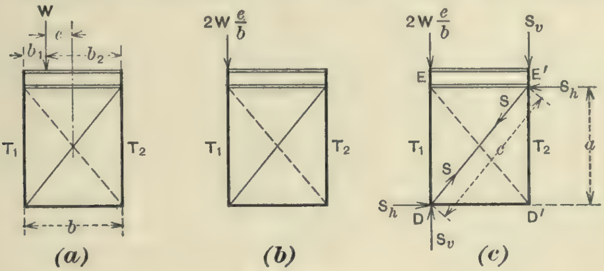
If complete systems of upper and lower lateral bracing are provided, the stresses in the sway bracing are statically indeterminate. In such cases it is often arbitrarily assumed that the sway bracing may transmit a horizontal half panel load, $\frac{1}{2}wp$, from the lower to the upper chord, inasmuch as the lateral truss in the plane of the latter is more rigid, especially when the bridge is not loaded vertically. It is not customary, however, to include such loads in computing the stresses in the upper lateral system. The stresses in the diagonal sway braces are usually so small that their exact value is a matter of little or no importance, since the size of these members is determined by practical rather than theoretical considerations.

The slope a/b of the diagonals should not exceed $1\frac{1}{2}$, or at the most, $1\frac{3}{4}$. For deep trusses the sway bracing should be arranged in double panels vertically, Fig. (*b*), or treble panels.

(*b*) *Through Bridges.* The depth of the sway bracing in through bridges is restricted by the clearance requirements underneath. The types of portal bracing treated in Arts. 240, 241, and 243 are also employed for sway bracing in through bridges, although much smaller sections are used for the individual members. Since the stresses in the sway bracing are always statically indeterminate, the sizes of the members are fixed according to judgment and successful precedent rather than on the basis of computed stresses. For deep trusses the sway bracing is usually of the double-diagonal type (Art. 240), arranged in one or more panels vertically, according to the depth of truss. The lattice type is also used, however.

247. Sway Bracing for Double-track Bridges. (*a*) *Deck Bridges.* Sway bracing is subject to stress from eccentric vertical loading, which becomes a factor of some importance in double-track bridges with two trusses when only one track is loaded. Single-track bridges are also subject to eccentric vertical loading from the effect of unbalanced locomotive mechanism, the unsymmetrical loading of freight cars, wind forces, and centrifugal forces (for bridges on curves).

Let Fig. 16 represent the cross-section of a double-track deck bridge, and W the live load on the floor beam when only one track is loaded. The corresponding joint loads for trusses T_1 and T_2 are Wb_2/b and Wb_1/b, respectively. The load on truss T_1 is greater by the amount $W(b_2-b_1)/b=2We/b$, where e denotes the eccentricity of the load. This excess load, Fig. (b), will cause truss T_1 to deflect vertically more than truss T_2, thus tending to twist the floor beam in the plane of the cross-section and subjecting its end connections to undesirable stresses. If it were practicable to transmit one-half of such excess loads to truss T_2 by means of the sway bracing, as specifications sometimes prescribe, both trusses would deflect equally under eccentric loading. This is not feasible, however. The proportion of excess load which may thus be transmitted increases with the rigidity (sectional area) of the diagonal braces, and the rigidity of the lateral systems and end bracing. Let S denote the stress in diagonal DE'. This stress cannot be developed if the lateral systems are not capable of transmitting the horizontal components S_h to the ends of the bridge, nor if the end braces are not capable of conveying the sum of the components in the plane of the upper chord to the masonry. The effect of the stress S is to reduce the panel load on truss T_1 by S_v, and to increase the panel load on truss T_2 by a like amount. The ratio of S_v



(a) (b) (c)

FIG. 16.

to $2We/b$ is statically indeterminate, and if the diagonal is proportioned on the assumption that its stress $S(=S_v\, c/a)$ is equal to $We/b \times c/a$, its actual stress for the assumed condition of eccentric loading will be considerably smaller. It is a good working rule, however, that the sway braces of double-track deck bridges shall be proportioned for a stress not less than

$$S = W\frac{e}{b}\cdot\frac{c}{d}, \quad . \quad . \quad . \quad . \qquad . \qquad (24)$$

where W is the maximum live floor-beam load for only one track loaded. It will often be found desirable, however, especially in the case of long-span bridges, to use larger sections, according to judgment, than those found by Eq. (24).

(*b*) *Through Bridges.* In through bridges the efficiency of the sway bracing in tending to equalize the loads on the trusses under eccentric loading is much lower than in deck bridges, because the sway bracing for the former does not extend over the full depth between the chords, and its lower connection points are much less rigid laterally. The dimensions of the sway bracing should be determined by judgment rather than computation, and it may consistently be made lighter for through than for deck bridges. Its purpose is to reduce vibrations rather than to equalize loads, and if it is designed for the minimum sizes and maximum slenderness ratios for compression members permitted by good specifications, it will usually be adequate for the purpose stated. The efficiency of the sway bracing depends, however, indirectly on the efficiency of the portal. In double-track through bridges with two trusses it is especially important, therefore, that the portal should be liberally proportioned.

248. Stresses in Main Trusses from Lateral Forces. Since maximum lateral forces in combination with maximum live loads are of rare occurrence the allowable unit stresses for that condition may properly be increased. Thus, the A. R. E. Specifications allow an increase of 25 per cent above that permitted for dead and live-load stresses only, while Cooper's Specifications allow an increase of 30 per cent above the *average* unit stress in the member from dead and live loading. Both methods will be treated in detail later (Art. 256).

Owing to the greater allowable unit stress in bridge members when the maximum stresses from the lateral forces are taken into account, these stresses need seldom be considered except in connection with the lower chord, for which the maximum tensile and compressive stresses from vertical and horizontal loading combined frequently demand attention. It is sometimes necessary to increase the sectional area, or to modify the form of cross-section, of the inclined end posts on account of the compressive stresses (direct and from bending) induced by the lateral forces.

(*a*) *Maximum Tension in Lower Chord.* The tensile stress from the lateral forces is greatest in the *leeward* chord, and it consists of the following three elements:

1. *The stresses in the members in their capacity as chord members in the lower lateral truss for maximum loading in the plane of that truss.* These stresses are found as in Art. 237.

2. *The stresses from the overturning effect of the wind pressure on the upper chord.* For present purposes it is sufficiently accurate to assume that none of this loading is transmitted to the lower chord through the sway bracing, but that it is first conveyed horizontally through the upper lateral bracing and then to the supports wholly through the portal or end bracing. Let w' denote the intensity of the lateral loading on the upper chord, and h the height of any panel point in the upper chord above the plane of the supports. The vertical reactions at the supports of the leeward truss will then be increased and those at the windward truss decreased, each by the amount

$$R_c = w'p/b \times \Sigma h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

the summation Σh extending from one end of the bridge to the center. If a joint occurs at the middle of the upper chord and the height of the truss at that point is h_1, $\frac{1}{2}h_1$ should be included in the summation. Suppose the lower chord to be horizontal, and let h_2 denote the height of truss at the hip joint. A change of R_c in the vertical reactions will then give rise to a change of $R_c\, p/h_2$ in the horizontal component of the stresses in the inclined end posts, and will therefore produce a change of that amount in the lower chord stresses throughout the bridge—tensile for the leeward truss, and compressive for the windward truss. For a truss with inclined chords, a rigorous analysis should take account also of the effect of the wind-load web stresses in the main trusses (Art. 237) on the lower chord stresses.

3. *The stresses from the overturning effect of the wind pressure on the train.* The latter may be assumed at 400 lbs. per linear foot of span, applied at a height of 7 ft. above the rail. If a denotes the height of this force above the plane of the *loaded* chord, b the width between centers of trusses, and p the panel length in feet, the effect will be the same as the addition of a vertical downward force of $400p \times a/b$ pounds at every joint of the *loaded* chord of the leeward truss, and a vertical upward force of a like amount at the corresponding joints of the windward truss. The tensile stresses in the leeward lower chord from these loads may be found in the usual way. In the case of a deck bridge the excess downward loading on the leeward truss will be transmitted in some measure—statically indeterminate (Art. 247)—to the windward truss, for which allowance should be made according to judgment. This excess load per joint is *twice* $400pa/b$. Again, for a deck bridge the stresses from the overturning effect of the wind pressure on the train, considered now to act in the plane of the upper chord, must be included. These stresses may be found as explained above in paragraph 2.

It is important to observe that in the above analysis the end posts are regarded as hinged at the base about horizontal axes in the planes of the trusses. In general, however, b denotes the height of any panel point in the upper chord *above the plane containing the sections of zero bending moment in the end posts* (Art. 239). The distance, y, in the direction of the end posts, from the bottom of the end posts to the section of zero bending moment may be found by Eq. (2), Art. 239, in which S_d is now the sum of the dead and live-load stresses in the end posts. If the value of y thus computed exceeds $\frac{1}{2}c$ (Fig. 3, Art. 239), it shows that S_d is large enough to ensure fixed-endedness at the base, and the latter value, $\frac{1}{2}c$, should then be used.

Considering now the combined tensile stresses in a given member of the leeward lower chord:

Let S_d = its dead-load stress;

S_l = its maximum live-load stress;

S_w = its maximum stress from the lateral forces, this being the sum of the stresses from the three elements just considered.

Then by the A. R. E. Specifications, S_w must be taken into account in proportioning the member if

$$S_w > 0.25(S_d + qS_l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or, by Cooper's Specifications, if

$$S_w > 0.30(S_d + S_l). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In (26), q is a factor greater than unity, intended to provide for impact. The methods of proportioning a member for combined dead, live, and wind-load stresses will be fully discussed later (Art. 256).

(*b*) *Maximum Compression in Lower Chord* (*Reversed Stresses*). The wind-load stresses in the lower chord of the *windward* truss are compressive. If these compressive stresses exceed the tensile stresses in that chord from vertical loading, the chord members should be designed so as to be capable of resisting the resultant compressive stress.

In that connection two conditions of loading should be considered:

First. The lateral forces acting on the unloaded bridge. The compressive stresses in the windward lower chord are then the sums of (1) the stresses in that chord acting as a chord of the lower lateral truss for the loads in the plane of that truss, and (2) the stresses from the overturning effect of the lateral forces on the upper chord. The resultant compressive stress, if any, is found by subtracting the dead-load tensile stress in each member from the sum of the above compressive stresses.

Second. The lateral forces acting on the bridge when loaded with a train of empty cars. (For a double-track bridge only the track towards the leeward truss should be assumed to be thus loaded.) For a wind pressure of 400 lbs. per foot applied 7 ft. above the rail, the empty cars should be assumed to weigh 1200 lbs. per linear foot (Art. 254). If the compressive stresses in the windward lower chord from the added horizontal load of 400 lbs. per foot exceed the tensile stresses from the added vertical load of 1200 lbs. per foot, this condition of loading will give greater resultant compressive stresses than the first condition, provided the assumed intensity of wind pressure is the same in both cases. Specifications frequently prescribe a greater intensity of wind pressure, however, for the unloaded than for the loaded structure, in which case the comparison must be made with that circumstance in mind.

As explained above in paragraph (*a*) 3, the horizontal load will produce an *upward* force of $400pa/b$ pounds at every windward joint of the loaded chord, while the vertical load will produce a *downward* force of $600p$ at these joints. Since $600p > 400pa/b$ these two vertical forces will produce *tension* in the windward lower chord. On the other hand, the horizontal loads, $400p$ pounds, at the windward joints will produce *compression* in the windward lower chord. Considering now a through bridge, if b denotes the width between centers of trusses and h the height of the main trusses, the resultant stress will be compressive if $400p/b > (600p - 400pa/b)/h$, i.e., when

$$\tfrac{2}{3}(a+h) > b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

For the ordinary relations of a, b, and h for single-track through bridges, criterion (28) will be satisfied, so that in general the second condition governs for maximum compression in the windward chord of through bridges.

In a similar analysis for deck bridges the considerations in paragraph (*a*) 3, applicable to deck bridges, must be kept in view. In this case, too, the second condition of loading will be found to govern for maximum compression in the windward lower chord.

For a bridge on a curve, the compressive stresses in the lower chord of the truss on the inner side of the curve are increased by the centrifugal force, and those stresses are therefore greatest when the wind force acts in the same direction as the centrifugal force. The resultant forces at the joints of the chord in question, from the vertical loading and the centrifugal force, may be found by Eq. (36), Art. 214, in which W is now $1200p$ pounds. The effect of the wind forces is the same as above, but the horizontal joint loads from wind must now be increased by those due to the centrifugal force.

In pin-connected bridges it is desirable to use eye-bars for the lower chord. If this chord is subject to small resultant compressive stresses, the eye-bars may be given some strength in compression by connecting the inner pair of bars by latticing. If, however, the compressive stresses are considerable, riveted compression members should be used instead of eye-bars. It is common practice to use such members in the end panels of bridges of moderate span for the sake of increased rigidity, irrespective of the computed stresses.

In general the compressive stresses in the lower chord may be reduced or entirely avoided by increasing the width between centers of trusses, or by decreasing the depth of the main trusses. The former involves added material in the floor beams (and possibly increased width of piers) and the latter added material in the trusses.

CHAPTER XV

STRESSES IN VIADUCT TOWERS

249. Definitions. Metal towers for the support of viaducts are not only much cheaper than masonry piers, but they can be carried to heights for which the latter are wholly unadapted. As ordinarily constructed, Fig. 1, each 'tower' consists of two vertical 'bents,' Fig. (*a*), connected by 'longitudinal bracing.' The bents are composed of a pair of vertical or inclined 'columns' or 'legs,' connected by 'transverse' or 'sway bracing.' The divisions of the tower are called 'panels' or 'stories.' For lateral stability against the overturning effect of wind pressure, or centrifugal forces in the case of railroad viaducts on curves, the columns are usually given a slope or 'batter' of 5 or 6 on 1 for single-track, and 8, 10 or 12 on 1 for double-track viaducts.

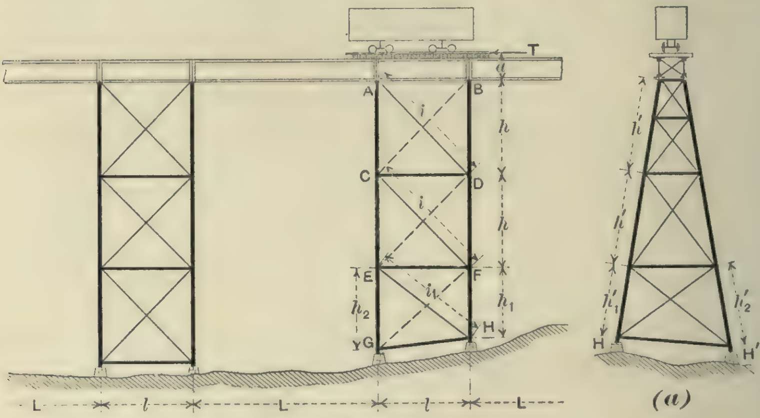


Fig. 1.

The economic considerations affecting the proportions and spacing of the towers will receive attention in Vol. III. At present only the analysis of the stresses will be considered, and the manner in which the stresses due to various influences should be combined in determining the governing conditions for each member.

250. Loads. The loads which have to be considered in the design of a railroad viaduct tower may be classified as follows:

I. Dead load: (*a*) Weight of superstructure.
(*b*) Weight of tower.
II. Live load: (*a*) Engines and train.
(*b*) Empty cars.
III. Wind load: (*a*) Wind pressure on live load.
(*b*) Wind pressure on superstructure.
(*c*) Wind pressure on tower.
IV. Tractive forces with brakes applied.
V. Centrifugal forces for viaducts on curves.

Snow load is rarely considered in the design of either railroad or highway viaducts, since maximum traffic cannot be maintained with any considerable amount of snow on the structure.

251. Stresses from Tractive Forces. According to the A.R.E. Specifications:

> "Viaduct towers and similar structures shall be designed for a longitudinal (tractive) force of 20 per cent of the live load applied at the top of the rail."

In practice each tower is usually assumed to transmit the tractive forces for the distance between centers of adjacent spans, i.e., the length $L+l$ (Fig. 1), and the tractive forces are usually computed on the basis of the uniform train load. Thus if T denotes the total tractive force which must be transmitted to the foundations along each inclined side of the tower, Fig. 1 (*a*), and if w denotes the uniform train load per foot of rail:

$$T=0.20w(L+l). \qquad \qquad (1)$$

Note. The factor 0.20 represents the coefficient of sliding friction on the rails, it being assumed that the efficiency of the brakes—i.e., the ratio of maximumpressure on brake-shoe to load on wheels—is sufficient to cause the wheels to 'skid' on the rails. In reality the efficiency of the brakes on freight and passenger cars is purposely made low enough to prevent skidding even when the cars are empty, so that the wheels will not be flattened. The efficiency of the brakes on locomotives is usually about 75 per cent of the weight of the loaded engine and 100 per cent of the weight of the unloaded tender. The maximum tractive force exerted on the rail by a given wheel is equal to the friction between the wheel and the brake-shoe under maximum pressure, provided no skidding occurs. The assumption underlying Eq. (1) is therefore more severe than the actual conditions warrant.* Tractive forces at the rail affecting the substructure can be developed only by the retardation or acceleration of the train as a whole.

The stresses in the longitudinal bracing are functions of the shear T, and are therefore not affected by the height a above the tower at which T may be assumed to be applied. For T acting in the direction shown, the stresses in the dotted diagonals are zero. The stresses in the other members are as follows:

$$CD(=EF) = -T;$$

$$AD(=CF) = +Ti/h';$$

$$EH = +Ti_1/h'_1.$$

in which i and i_1 denote the true lengths of the diagonals which lie in inclined planes. If the bases, H and H', of the bent are not on the same level, as in Fig. (a), the stress in the diagonal $E'H'$ (not shown) on the opposite side of the tower from EH is

$$E'H' = +Ti_2/h'_2.$$

The stress in the bottom strut GH depends on the conditions at the supports. If H is a 'fixed' or 'sliding' support, stress GH is zero for T acting in the direction shown, since the sliding friction at the base will exceed that on the rail. If H is a 'roller' support the horizontal component of stress GH is T minus the rolling friction at H. In practice this strut is usually given the same cross-section as struts CD and EF. It should be sufficiently strong in tension and compression to overcome the friction at the movable support for the unloaded structure, so as to ensure movement at this point under changes of temperature. If R_d is the dead-load reaction at the base of the column and c the coefficient of friction, the requisite stress, if GH is horizontal, is

$$GH = \pm cR_d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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[*] *Traction Stresses*, by S. Blumenthal, Proc. Can. Soc. C. E., Vol. XXIV, 1910.

in which the value of c may be assumed at 0.02 for rolling friction, 0.2 for sliding friction between planed plates, and 0.3 for sliding friction between unplaned plates. If the strut is inclined, as in the figure, Eq. (2) denotes the horizontal component of the stress.

The stresses in the diagonals produce cumulative stresses in the columns. Thus

$$\left.\begin{aligned} AC &= -Th'/l \\ CE &= -2Th'/l \\ EG &= -(2Th'/l+Th'_1/l) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)a$$

$$\left.\begin{aligned} BD &= 0 \\ DF &= +Th'/l \\ FH &= +2Th'/l \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)b$$

The stresses in the columns are affected by an equal amount throughout each column by any disturbances produced by T in the vertical reactions at A and B. If the tractive force is assumed to be uniformly distributed along the head of the rail, as in the A.R.E. Specifications, it will produce no effect on the vertical reactions at A and B. Thus considering the effect at A, for T acting in the direction shown, the traction for the span L to the left will develop a negative reaction $0.20wL\times a/L(=0.20aw)$ at A, and the traction for the span l to the right will develop a positive reaction, $0.20wl\times a/l(=0.20aw)$ at A, so that the net reaction at A is zero. Strictly speaking, if the brakes are applied to a loaded car moving under its own momentum, that is, without tension in the draw-bar, the tractive force at the head of the rail is one force of a couple of which the other force is at the center of gravity of the loaded car, i.e., at a height b above the head of the rail. If the weight of the car is W, and the distance between truck centers is c, the reaction at one truck will be increased and at the other decreased by the amount $0.20Wb/c$. The resultant effect on the reactions at A and B will depend on the relative positions of the various truck centers on the adjacent spans, the negative and positive reactions tending, however, to neutralize each other, as explained above for truly uniform loading. It is sufficiently accurate, therefore, for practical purposes to assume that the tractive forces do not affect the reactions at A and B, which is equivalent to assuming T applied at the level of A–B.

SYMMETRICAL BENT

252. Stresses from Dead Load. Referring to the bent shown in Fig. 2, let W equal the loads at A and A' from the weight of the superstructure, including the girders, or trusses, and the track; and let W_1 equal the weight per panel of each column, and the bracing in both directions tributary to each tower apex. The stresses in the columns are then

$$\left.\begin{aligned} AB(=A'B') &= -(W+0.5W_1)h'/h \\ BC(=B'C') &= -(AB+W_1h'/h) \\ CD(=C'D') &= -(BC+W_1h'/h) \end{aligned}\right\} . \quad . \quad . \quad . \quad (4)$$

If the diagonals are flexible members incapable of acting in compression, as may usually be assumed without important error, symmetrical vertical loads can produce no stresses in these members. This may be seen by taking a horizontal section through any panel and noting that the resultant moment of the forces above or below such a section about the moment center at the intersection of the columns produced, is zero. If the diagonals are capable of developing compression they will be subject to compressive stresses, for as the columns are shortened through their elastic deformation, the diagonals are also shortened. Compressive stresses are thus induced in these members and tensile stresses in the struts BB' and CC'. These stresses as well as the stresses in the columns are then statically indeterminate. Sometimes the diagonals are made 'stiff,' i.e., capable of acting in compression, and the intermediate horizontal struts are omitted.

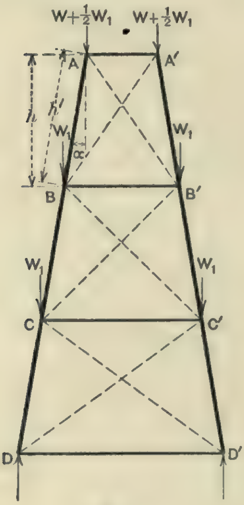

FIG. 2.

In that case under vertical loading the horizontal components of the stresses in the diagonals will produce outward thrusts at B, B', C, and C', that is, bending in the columns.

If the stresses in the diagonals are assumed to be zero, the stresses in the struts may be found by applying $\Sigma H=0$ successively to the forces at A, B, C and D. Thus

$$AA' = -(W+\tfrac{1}{2}W_1)\tan\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$BB'(=CC') = -W_1 \tan\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As previously stated, for single-track viaducts the value of $\tan\alpha$ is usually $\frac{1}{5}$ or $\frac{1}{6}$, and for double-track viaducts, $\frac{1}{10}$ or $\frac{1}{12}$.

The stress in the bottom strut will be considered later (Art. 258).

253. Stresses from Live Load. Referring to Fig. 2 of the preceding article, let W_l denote the maximum load at A and A' from the live loading on the spans L and l adjacent to the bent. This may be found as explained in Art. 45. The maximum compressive stresses for both columns are then

$$AD(=A'D') = -W_l h'/h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The stress in AA' from the loads W_l may be found as in the last article, i.e.,

$$AA' = -W_l \tan\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The loads W_l do not affect the stresses in the intermediate struts. The stress in the bottom strut will be considered later (Art. 258).

The resultant wind-load stresses in the windward column are sometimes tensile, as will be seen in the next article. In the analysis of these stresses the effect of the wind pressure on a train of empty cars must be considered. If W_e denotes the resulting load at A and A',

$$AD(=A'D') = -W_e h'/h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

254. Stresses from Wind Load. According to the A.R.E. Specifications:

"Viaduct towers shall be designed for (1) a wind force of 50 lbs. per sq. ft. on one and one-half times the vertical projection of the structure unloaded; or 30 lbs. per sq. ft. on the same surface, plus 400 lbs. per linear ft. of structure applied 7 ft. above the rail for assumed wind force on train when the structure is either (2) fully loaded, or (3) loaded on either track with empty cars assumed to weigh 1200 lbs. per linear ft., whichever gives the larger stress."

The above conditions of loading will be referred to hereafter as Cases (1), (2) and (3).

In a standard-gauge track the rails are approximately 5 ft. center to center. A wind pressure of 400 lbs. per foot of structure applied 7 ft. above the rail will therefore overturn a car weighing less than $(400 \times 7)/2.5 = 1120$ lbs. per foot of length.

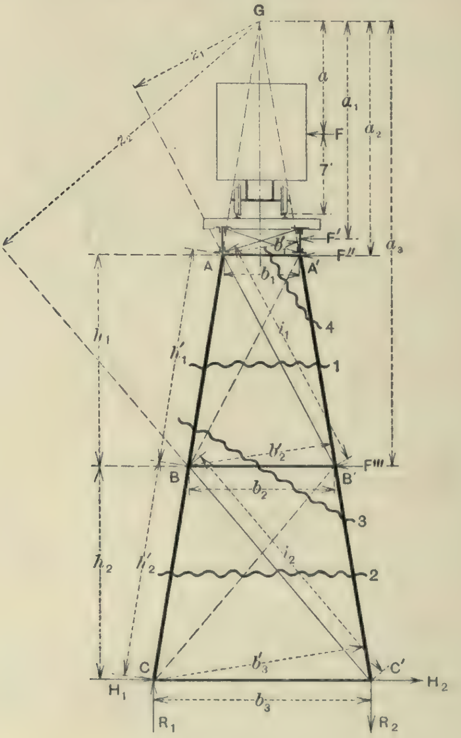

FIG. 3.

Referring to Fig. 3, let F denote the total wind load on the train for the distance between centers of the spans adjacent to the bent, i.e., $\frac{1}{2}L + \frac{1}{2}l$ (Fig. 1). Hence for Cases (2) and (3), $F = 200(L + l)$. Let F' denote the total wind load on the superstructure for the distance $\frac{1}{2}(L+l)$, found by multiplying one and one-half times the vertical projection, in square feet, of the superstructure for that distance, by 50 for Case (1), or 30 for Cases (2) and (3). Assume F' applied at the level of the center of gravity of the vertical projection of the superstructure. Finally, let F'' and F''' denote the total wind loads on the parts of the substructure tributary to apexes A' and B' respectively, found by multiplying one and one-half times the vertical projections, in square feet, of the corresponding parts of the bent and longitudinal bracing by 50 for Case (1), or 30 for Cases (2) and (3). It is sufficiently accurate for practical purposes to assume F'' and F''' to be applied wholly at the windward apexes instead of partly at the leeward apexes A and B, respectively. The slight errors are on the side of safety and affect only the stresses in the struts AA' and BB'.

If the intermediate horizontal struts are omitted the wind forces on the bent are usually assumed to be applied in equal parts to the two columns. This leads to equal (+ and −) stresses in the diagonals in the same panel.

The symbols F'_{30}, F''_{30}, F'''_{30}, and F'_{50}, F''_{50}, F'''_{50} will be used to indicate that the loads are based on assumed pressures of 30 and 50 lbs. per square foot, respectively.

(*a*) *Diagonals.* It will be assumed that the diagonals are flexible members incapable of acting in compression. Then for the wind in the direction shown, the stresses in the dotted diagonals are zero. It has been seen that the vertical loads produce no stresses in the diagonals (Art. 252). Taking sections 1 and 2, and considering in turn moments about G of the forces above these sections:

Case (1),

$$\left.\begin{aligned} AB' &= (+)[(F'_{50}a_1 + F''_{50}a_2) = M_{50}] \div z_1 \\ BC' &= (+)[(M_{50} + F'''_{50}a_3) = M'_{50}] \div z_2 \end{aligned}\right\} \quad . \; . \; . \; . \quad (10)$$

Cases (2) and (3),

$$\left.\begin{aligned} AB' &= (+)[(Fa + F'_{30}a_1 + F''_{30}a_2) = M_{30}] \div z_1 \\ BC' &= (+)[(M_{30} + F'''_{30}a_3) = M'_{30}] \div z_2 \end{aligned}\right\}, \quad . \quad (11)$$

in which by Eq. (12), Art. 131, $z_1 = a_2b_2/i$, and $z_2 = a_3b_3/i_2$. These lever arms may also be determined with sufficient accuracy from a drawing constructed to a suitable scale. For each member the greater stress from Eqs. (10) and (11) must be used. The lever arms of the diagonals increase much more rapidly for successive panels downward than the moments about G which remain nearly constant. The stresses in the diagonals therefore decrease rapidly for successive panels downward.

(*b*) *Struts*. For BB', taking section 3 and moments about G, Case (1),

$$BB' = -M'_{50} \div a_3. \qquad \qquad (12)$$

Cases (2) and (3),

$$BB' = -M'_{30} \div a_3, \qquad \qquad (13)$$

in which M'_{50} and M'_{30} have the values expressed in Eqs. (10) and (11). The greater stress from Eqs. (12) and (13) must be used.

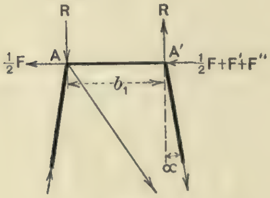

FIG. 4.

Considering the top strut, AA', it has been seen that this member is subject to compression from the live load, (Eq. (8), Art. 253). Therefore Case (2), for maximum live loads W_l at A and A', is the governing case. If c and c_1 denote respectively the vertical distances from AA' to F and F', the vertical reactions R, Fig. 4, produced by these forces at A and A', are $(Fc+F'c_1)/b_1$ and F and F' may now be considered applied at the level A–A'. That is, the original forces F and F' are equivalent to the couple Rb_1 and the force $F+F'$ applied along AA', as in Fig. 3, Art. 15. The proportions in which $F+F'$ is divided between A and A' are statically indeterminate. Reverting to Fig. 3, it may reasonably be assumed that F is carried in equal parts to A and A' and that F' is conveyed wholly to A'. The forces at A and A' are then as shown in Fig. 4. From $\Sigma H=0$ applied to A',

$$AA' = (-)(\tfrac{1}{2}F + F' + F'' - R \tan \alpha), \qquad (14)$$

in which $R = (Fc+F'c_1)/b_1$.

The stress in the bottom strut will be considered later (Art. 258).

(*c*) *Compression in Columns*. Since the compression in the leeward column AC, Fig. 3, is evidently maximum when the structure is fully loaded, only Case (2) need be considered. Take sections 1 and 2 and moments about B' and C', respectively. Let $M_{30}^{B'}$ denote the moment of F, F'_{30} and F''_{30} about B', and $M_{30}^{C'}$ the moment of these forces and F'''_{30} about C'. Then

$$\left.\begin{aligned} AB &= -M_{30}^{B'} \div b'_2 \\ BC &= -M_{30}^{C'} \div b'_3 \end{aligned}\right\}, \quad \ldots\ldots \tag{15}$$

in which $b'_2 = b_2 h_1/h'_1$, and $b'_3 = b_3 h_1/h'_1$.

(*d*) *Tension in Columns.* For maximum tension in the windward column $A'C'$, Cases (1) and (3) must be considered. The moment centers of $A'B'$ and $B'C'$ lie at A and B respectively. Let M_{50}^{A} denote the moment of F'_{50}, and M_{30}^{A} the moment of F and F'_{30} about A. Then

$$A'B' = +M_{50}^{A} \div b'_1, \quad \ldots\ldots \tag{16}$$

or

$$A'B' = +M_{30}^{A} \div b'_1. \quad \ldots\ldots \tag{17}$$

Similarly, if M_{50}^{B} denotes the moment of F'_{50} and F''_{50}, and M_{30}^{B} the moment of F, F'_{30} and F''_{30} about B,

$$B'C' = +M_{50}^{B} \div b'_2, \quad \ldots\ldots \tag{18}$$

or

$$B'C' = +M_{30}^{B} \div b'_2. \quad \ldots\ldots \tag{19}$$

The wind-load tensile stresses in Eqs. (17) and (19) must be combined with the live-load compressive stresses from a train of empty cars, by Eq. (9), Art. 253. Thus, letting h'/h denote $h_1'/h_1 = h_2'/h_2$,

$$A'B' = (+)\left(\frac{M_{30}^{A}}{b'_1} - \frac{W_e\, h'}{h}\right), \quad \ldots\ldots \tag{20}$$

and

$$B'C' = (+)\left(\frac{M_{30}^{B}}{b'_2} - \frac{W_e\, h'}{h}\right). \quad \ldots\ldots \tag{21}$$

Since $b'_1 = b_1 h/h'$ and $b'_2 = b_2\, h/h'$,

$$A'B' = (+)\left(\frac{M_{30}^{A}}{b_1} - W_e\right)\frac{h'}{h}, \quad \ldots\ldots \tag{22}$$

$$B'C' = (+)\left(\frac{M_{30}^{B}}{b_2} - W_e\right)\frac{h'}{h}. \quad \ldots\ldots \tag{23}$$

For $A'B'$ the greater stress from Eqs. (16) and (22) and for $B'C'$ the greater stress from Eqs. (18) and (23) must be used. If in Eqs. (22) and (23) W_e is greater than the first term in the parentheses, the resultant stress is compressive.

255. Stresses from Centrifugal Forces. According to the A.R.E. Specifications:

> "The centrifugal force shall be considered as live load and be derived from the speed in miles per hour given by the expression $60-2\frac{1}{2}D$, where D = degree of curve."

The values of the centrifugal force, F_c, in terms of the vertical load, W, and for different speeds, $V(=60-2\frac{1}{2}D)$, are given in the table in Art. 212. In the present case W is the maximum total live-load reaction on the bent. Whenever the centrifugal force tends to increase the stress in a member, assume the load moving at full speed, and vice versa. The centrifugal force, F_c, should be assumed to act at the level of the center of gravity of the live load, i.e., about 5 ft. above the rail. The stresses from this force in the columns and transverse bracing may be found by moments in precisely the same way as the stresses from the wind force F, Fig. 3, were derived in the preceding article.

256. Required Sectional Area for Combined Stresses. Various methods of combining dead and live-load stresses in proportioning truss members have been considered in Art. 102. Since the maximum stresses from wind and traction are rarely developed, it is customary to increase the allowable unit stress in considering these stresses in combination with the dead and live-load stresses. Thus the A.R.E. Specifications allow an increase of 25 per cent above the unit stress allowed in a member proportioned for dead and live-load stresses only. The contingency that maximum live-load, wind-load and traction stresses will be developed concurrently, is, however, so remote that this combination is usually disregarded, as will be done here. Since centrifugal forces are developed at every passage of a train on a curve, the stresses from this cause should be treated in exactly the same manner as the live-load stresses. Unless otherwise stated, it is to be understood, therefore, that any stresses from centrifugal forces are to be regarded as included with the live-load stresses in what follows.

Let S_d = dead-load stress in a member;
S_l = live-load stress in a member;
S_w = wind-load stress in a member;
S_t = tractive-load stress in a member;
s = allowable unit stress for S_d, and for S_l increased for impact;
s_1 = allowable unit stress for S_d, S_l and S_w (or S_t);
a = required sectional area for S_d and S_l;
a_1 = required sectional area for S_d, S_l and S_w (or S_t).

(*a*) *By A. R. E. Specifications.* Assume the stresses in the member under consideration to be of the same character, and let q denote the *impact factor* for transforming the live-load stress into an 'equivalent' dead-load stress. Then

$$a = (S_d + qS_l) \div s. \qquad \ldots \ldots \qquad (24)$$

By the A.R.E. Specifications, $s_1 = 1.25s$, whence

$$a_1 = [S_d + qS_l + S_w (\text{or } S_t)] \div 1.25s. \qquad \ldots \qquad (25)$$

If S_w (or S_t) $> 0.25(S_d + qS_l)$, $a_1 > a$, and vice versa.

By Art. 102 (First Method), $q = 1 + k$, in which k is the percentage added to S_l to allow for impact.

(*b*) *By Cooper's Specifications* the allowable unit stress for dead-load stresses is twice as great as for live-load stresses, as in Art. 102 (Second Method). If wind stresses are included, an increase of 30 per cent is allowed above the *average* unit stress from the dead and live-load stresses.

Thus, if s_d = allowable unit stress for S_d,
$\frac{1}{2}s_d$ = allowable unit stress for S_l.

Hence
$$a = S_d/s_d + S_l/\tfrac{1}{2}s_d,$$
or
$$a = (S_d + 2S_l) \div s_d, \qquad \ldots \ldots \qquad (26)$$
and
$$s_1 = 1.30(S_d + S_l) \div a.$$

Finally,
$$a_1 = \frac{S_d + S_l + S_w \text{ (or } S_t)}{s_1} = \frac{S_d + S_l + S_w \text{ (or } S_t)}{1.30(S_d + S_l)} a \quad . \qquad (27)$$

The required sectional area is determined by Eq. (26) or (27), according to which gives the greater value. Comparing these equations, it is seen that when S_w (or S_t) $> 0.30(S_d+S_l)$, $a_1 > a$, and vice versa.

Of the above two methods the one prescribed by the A.R.E. Specifications may be regarded as more rational, since the value S_l is there consistently weighted in the expression for s_1.

257. Governing Combinations of Stresses. (*a*) *Longitudinal Bracing.* The longitudinal bracing is assumed to be affected only by the tractive forces, for which the stresses were derived in Art. 251. These stresses are much higher than those from wind in the direction of the viaduct, a condition which has not been considered, since the combination of such stresses with those from traction may be dismissed as too improbable.

(*b*) *Transverse Diagonals.* These must be proportioned for the wind-load stresses by Eqs. (10) or (11), Art. 254. In the case of viaducts on curves the stresses from the centrifugal forces must be added for those diagonals for which these stresses are tensile. For the other diagonals the train may be regarded as moving slowly, or at rest.

(*c*) *Intermediate Transverse Struts.* These are subject to the dead-load stress given by Eq. (6), Art. 252, and the wind-load stresses given by Eqs. (12) or (13), Art. 254. The dead-load stress is relatively so small that it is frequently neglected. For viaducts on curves the stresses from the centrifugal forces must be added.

(*d*) *Top Transverse Strut.* This member must be proportioned according to Eq. (24) or (26) of the preceding article, in which S_d is found by Eq. (5), Art. 252, and S_l by Eq. (8), Art. 253. If S_w from Eq. (14), Art. 254, exceeds 0.25 (S_d+S_l), the sectional area must be increased according to Eq. (25) or (27) of the preceding article. For viaducts on curves the stresses from the centrifugal forces must be added. The centrifugal force, F_c, should be treated the same as F in Eq. (14), Art. 254.

If the resultant stress in the topmost section of the column, under the combined influence of wind and centrifugal forces, should be tensile, which is seldom or never true, the stress in the top transverse strut will be tensile if the horizontal component of the stress in the column exceeds the sum of the horizontal forces to the right of section 4, Fig. 3, Art. 254, as may be seen from $\Sigma H=0$ applied to these forces. Under ordinary conditions this member is not subject to tensile stress.

(*e*) *Compression in Columns.* The columns must be proportioned for compressive stresses by Eq. (24) or (26) of the preceding article (in which s varies according to the slenderness ratio of the column), S_d being found by Eq. (4), Art. 252, and S_l by Eq. (7), Art. 253. If, however, S_t by Eqs. (3)a, Art. 251, or S_w by Eqs. (15), Art. 254, exceed $0.25\ (S_d+qS_l)$, the sectional area must be increased according to Eq. (25) or (27) of the preceding article.

(*f*) *Tension in Columns.* The resultant stresses in the lower columns of high towers may be tensile under the influence of either wind or traction,

(1) *From Wind Load.* The tensile stresses by Eqs. (16) and (18), or Eqs. (22) and (23), Art. 254, must be combined with the dead-load compressive stresses by Eq. (4), Art. 252. For maximum tension from Eqs. (22) and (23) the train of empty cars should be assumed to be moving slowly, so that the negative term W_e need not be increased for impact (Art. 256).

(2) *From Traction.* The tensile stresses by Eq. (3) b, Art. 251, must be combined with the compressive stresses from the dead load and from the live load used in computing the traction, i.e., a uniform train load of w per foot of rail. The compressive stress in the columns from this load is $\frac{1}{2}w(L+l)h'/h$. The impact factor q is omitted for reasons stated under (1).

If the tractive force T is assumed applied at the level of the top of the tower (Art. 251), then for a coefficient of sliding friction of 0.20, the slope of the resultant, R, of T and the weight of the live load, W_l, per rail, per tower, is 5 on 1 as in Fig. 5. Taking section 1, R lies to the right of C, the moment center of $C'D'$; but to the left of D, the moment center of $D'E'$. The resultant stress from traction and live load is therefore compressive for $C'D'$ and tensile for $D'E'$. The latter must be combined with the dead-load compressive stress in that member.

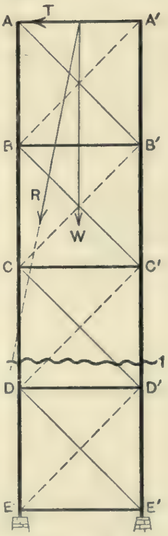

FIG. 5.

The possible governing combinations of stresses may be conveniently tabulated as follows, the asterisks in the same horizontal line denoting the stresses to be combined:

GOVERNING COMBINATIONS OF STRESSES

Members.			Dead, (S_d)	Live, (qS_l)		Wind, (S_w)		Traction, (S_t)	Centrifugal.
				Max.	Empty Cars.	Loaded, 30 lbs. per sq.ft.	Unloaded, 50 lbs. per sq.ft.		
Columns	Compr.	I	*	*	......	......	......	......	*
		II[1]	*	*	......	*	......	......	*
		III[2]	*	*	......	......	......	*	*
	Tension	I	*	......	*[4]	*	......	......	*
		II	*	......	......	......	*	......	......
		III	*	*	......	......	......	*	*
Transverse bracing	Diagonals	I[3]	......	......	......	*	......	......	*
		II	......	......	......	......	*	......	......
	Interm. struts	I	* (neglect)	......	......	*	......	......	*
		II	*	......	......	......	*	......	......
	Top strut	I	*	*	......	......	......	......	*
		II[1]	*	*	......	*	......	......	*

[1] For $S_w > \frac{1}{4}(S_d+qS_l)$.

[2] For $S_t > \frac{1}{4}(S_d+qS_l)$.

[3] See paragraph (*b*) above.

[4] For this case use S_l instead of qS_l.

258. Stresses in Bottom Transverse Strut. This member, *FF′*, Fig. 6, does not come into play until the resultant of the horizontal components of *EF* and *E′F*, which meet at *F*, exceeds the frictional resistance at that support, or a similar condition obtains at the other base, *F′*.

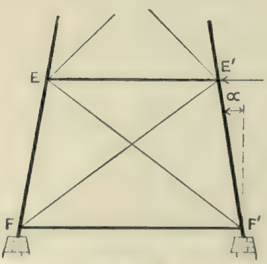

FIG. 6

If with the wind in the direction of the arrow, the resultant stress in *E′F′* by (*f*) (1) of the preceding article is tensile, the reaction at *F′* is negative, and the horizontal components of *EF′* and *E′F′* will both tend to cause compression in *FF′*. The anchor bolts at *F′* are then in tension, and under proper adjustment the frictional resistance at *F′* is developed between the nuts at the top of these bolts and their bearing on the pedestal. This frictional resistance must be overcome before *FF′* receives compression.

Again, under maximum compression in EF and $E'F'$, let R denote the reaction at each base. Let $\tan \alpha = \frac{1}{6}$, and assume a sliding support at F or F' and the coefficient of friction $c = \frac{1}{5}$. The horizontal component, $\frac{1}{6}R$, of stress EF or $E'F'$ will then be insufficient to overcome the friction $\frac{1}{5}R$, and stress FF' is zero.

The bottom strut should be sufficiently strong to ensure movement at the sliding or rolling support in the *unloaded* structure, under changes of length from variations in temperature. Thus for the above assumed conditions, and letting R_d denote the dead-load reaction at each base, the stress under rising temperatures will be

$$FF' = (-)\left(\tfrac{1}{5} - \tfrac{1}{6}\right)R_d,$$

and under falling temperatures,

$$FF' = (+)\left(\tfrac{1}{5} + \tfrac{1}{6}\right)R_d.$$

In practice the bottom strut is frequently designed for a tensile stress of $R \tan \alpha$, in which R denotes the maximum reaction at each base from the dead and live load, the effect of friction being disregarded. The member is then proportioned as a strut with a slenderness ratio sufficiently low to ensure its effective action in compression under rising temperatures.

259. Stresses in Anchorage. The analysis for maximum tension in the anchorages at the column bases must be made for the same conditions as for maximum tension in the columns, Art. 257 (*f*), the center of moments being assumed at the opposite base. Moderate negative reactions at the bases under extreme conditions are unobjectionable if the anchorages are properly designed for their development.

260. Graphic Methods. The analysis of the stresses should be planned with careful regard to the requirements tabulated under "Governing Combinations of Stresses" in Art. 257. The stresses from the vertical loads and traction may be most readily found by computation.

The stresses from the wind and centrifugal forces may be conveniently found graphically, if preferred, by constructing two diagrams, the first for the stresses from F, the wind load of 400 lbs. per linear foot of structure, applied 7 ft. above the rail, and the second for the wind load on the structure.

Since F is assumed to be carried in equal parts to A and A', Fig. 3, the forces at these points are as shown in Fig. 7(a), and the stress diagram may be constructed in the usual manner, by beginning with the forces at A'. The resultants R and R' of the external, forces at A and A', respectively, have the same slope, and therefore intersect F as shown in Fig. (b). OA and OA' may be considered false members, and the stress diagram may then be drawn by beginning with the forces at O.

The second diagram may be drawn for a wind pressure of 30 lbs. per square foot on one and one-half times the vertical projection of the structure unloaded, the resulting wind forces being assumed applied wholly at the windward side. The resultant wind force, F', on the superstructure may be treated as in Fig. 7 (a), or as in Fig. (b), AO being now drawn vertically from A to its intersection with F'. The stresses from a wind pressure of 50 lbs. per square foot may be found by multiplying the stresses from the diagram by 5/3.

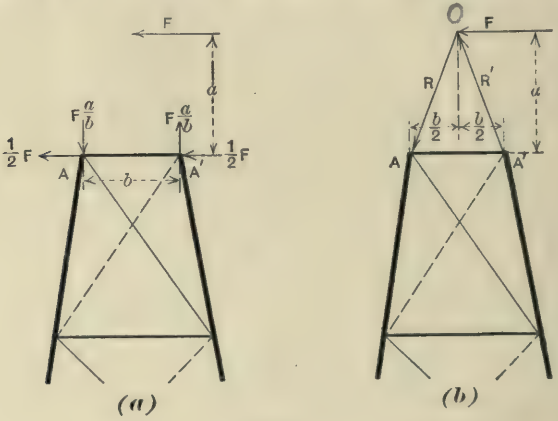

FIG. 7.

The stresses from centrifugal forces may be found in precisely the same manner as the wind-load stresses in Fig. 7.

UNSYMMETRICAL BENT, OR UNSYMMETRICAL VERTICAL LOAD

261. Viaducts on Curves. For viaducts on curves the columns on the outer side of the curve are sometimes given an increased batter, as shown in Fig. 8, so as to make the line of action of the resultant of all forces that may be developed simultaneously fall within the distance CC' between the bases, thereby avoiding negative reactions at C'.

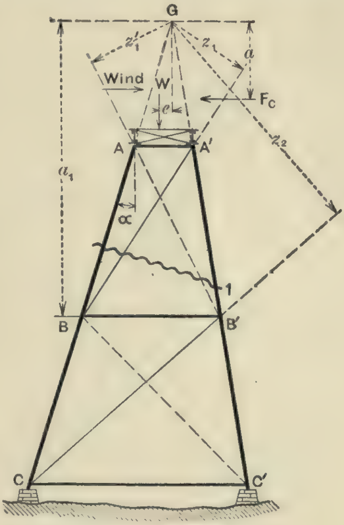

Fig. 8.

Transverse Bracing. Let W denote the resultant of the dead load and the maximum live load on the bent. Taking section 1, the moment center of $A'B$ lies at G, and since W has an eccentricity of e with respect to a vertical through G, the diagonals $A'B$ and $B'C$ will be subject to tensile stresses from the vertical loading. Thus

$$\left.\begin{aligned} A'B &= +We/z_1 \\ B'C &= +We/z_2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (28)$$

Similarly,

$$BB' = -We/a_1 \quad \ldots\ldots\ldots \quad (29)$$

If R_A = reaction at A from W,

$$AA' = -R_A \tan\alpha. \quad \ldots\ldots\ldots \quad (30)$$

For maximum resultant stresses two cases must be considered. Let W_d and W_l denote, respectively, the dead- and live-load elements of W, i.e., $W_d + W_l = W$.

Case I. The stresses from Eqs. (28), (29), and (30), combined with those from wind under the assumed maximum pressure for the *loaded* structure, acting in the direction of the arrow on a slowly moving train, so that the centrifugal force, which is exerted in the opposite direction, may be negligible. In that case W_l should include no allowance for impact.

Case II. The same conditions as for Case I, except that the live load is assumed to move at full speed. The effect of the centrifugal force must then be taken into account, and both F_c and W_l must include the same proportionate increase for impact.

If an impact factor q is applied to W_l, the center of gravity of W_l and W_d will be shifted, thus affecting the eccentricity e in Eqs. (28), (29), and (30). It is best therefore to compute the dead and live-load stresses separately.

For maximum stresses in AB' and BC' several conditions must be considered.

Case I. Assume the wind to act in the opposite direction from that indicated in the figure, and the train to move at full speed for maximum centrifugal force. Observe that the tensile stresses in $A'B$ and $B'C$ must be neutralized before AB' and BC' will come into play. Thus if M_G is the counter-clockwise moment about G of the centrifugal and wind forces above section 1, and S is the tensile stress in $A'B$ from the vertical loads,

$$AB' = (+)(M_G - Sz_1)/z_1' \quad . \quad . \quad . \quad . \quad . \quad (31)$$

in which $Sz_1 = We$. That part of M_G due to F_c, and that part of S due to W_l, should be increased for impact. Stress BC' may be found similarly.

In Fig. 8, let F_c denote the centrifugal force from the maximum live load included in W. The force F_c tends to develop a tensile stress $F_c a/z_1'$ in AB', and W tends to develop a compressive stress We/z_1' in that member. If $We > F_c a$ the resultant tendency of the live load is to develop compression in AB'. Since that member is incapable of acting in compression the effect will be a tensile stress in $A'B$. Under these circumstances Case I will not govern for stresses AB' and BC'.

Case II. If $We > F_c a$ the live load should be assumed to consist of a train of empty cars, weighing 1200 lbs. per foot, moving at full speed under the assumed maximum wind pressure for the *loaded* structure. Eq. (31) is applicable to this case by assigning values to M_G and S (including impact) corresponding to this new condition of loading.

Case III. The effect of maximum wind pressure on the *unloaded* structure must be considered, the intensity of this pressure being assumed at five-thirds of that for the loaded structure (Art. 254). If M_w is the moment about G of the wind forces above section 1, and S_d the dead-load tensile stress in $A'B$,

$$AB' = +(M_w - S_d z_1)/z_1' . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If $S_d z_1 > M_w$, AB' cannot act in tension for this loading, and if also for Case II, $Sz_1 > M_G$ in Eq. (31), only a single tension diagonal $A'B$ need be provided.

Columns. For maximum compression in columns, for AB and BC consider full loading at full speed, and for $A'B'$ and $B'C'$ full loading at slow speed.

262. Double-track Viaducts. In a symmetrical bent eccentric vertical loads will produce stresses in the transverse bracing. Thus let Fig. 9 represent one bent of a double-track viaduct, and let W denote the maximum live load that can be transmitted to the bent from one track, the other track being assumed to be unloaded. This load will produce tensile stresses in the diagonals represented by full lines, and compressive stresses in the struts. Thus, by moments about G,

$$\left.\begin{aligned} AB' &= +We/z_1 \\ BC' &= +We/z_2 \\ BB' &= -We/c \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

From $\Sigma H = 0$ applied at A', stress AA' is equal to the horizontal component of stress $A'B'$ whose vertical component is the reaction Wa/b at A' from W. Hence

$$AA' = -W\frac{a}{b}\tan\alpha . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The live-load stress in AA' becomes maximum, however, when both tracks are fully loaded. The reactions at A and A' are then W, and the stresses in the flexible diagonals and in BB' are zero. For that loading

$$AA' = -W\tan\alpha . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

For double-track viaducts $\tan\alpha$ is usually $\frac{1}{8}$, $\frac{1}{10}$ or $\frac{1}{12}$.

The principal function of AA' is its action as a girder in supporting the intermediate longitudinal track girders. Member AA' is therefore designed as a girder and its lower flange is in tension under vertical loading. Since the stress from Eq. (35) is compressive it tends to reduce the tensile stress in the lower flange. In practice the stress from Eq. (35) is therefore usually neglected and AA' is designed merely as a girder for vertical loading.

For maximum stresses in the transverse bracing the stresses from Eq. (33) must be combined with the wind-load stresses, and (for the struts) with the dead-load stresses, as in Art. 257. The stress from Eq. (35) must be combined with the dead-load stress. The wind-load stress must also be included when it exceeds one-fourth of the sum of the dead and live-load stresses, after the latter have been increased for impact (Art. 256).

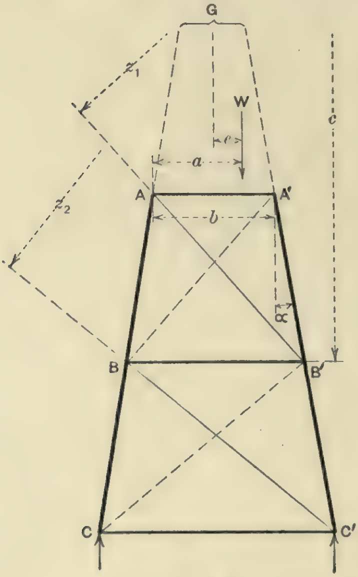

FIG. 9.

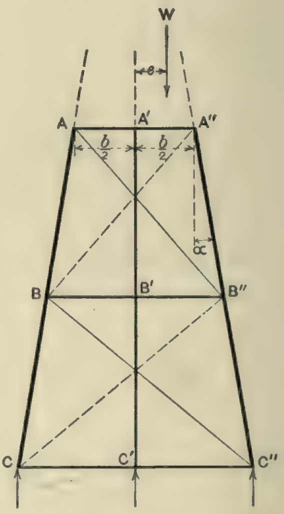

FIG. 10.

Double-track viaducts are sometimes supported on three columns, as in Fig. 10.* The above analysis of the stresses in the bracing applies to this case without change, except with respect to the top strut. The stress in that strut becomes maximum when one or both tracks are fully loaded. Thus for the right track fully loaded, as in the figure, the reaction at A'' is $We/\frac{1}{2}b$ and

---

* In such a design longitudinal bracing in three planes should be provided. If the central columns are connected with the transverse bracing, as is neces-

$$AA'' = -W\frac{e}{\frac{1}{2}b}\tan\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

which is less than the value by Eq. (34). Since a load between A and A' will not affect the reaction at A'', stress AA'' will remain unaltered, so long as the reaction at A does not exceed that at A'', which would bring the diagonal $A''B$ into play instead of AB'', thereby affecting the forces which meet at A'', including AA''. For both tracks fully loaded, stress AA'' is therefore the same as in Eq. (36).

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sary to keep their slenderness ratios within economic limits, their elastic deformation under vertical loads will induce stresses in the transverse bracing, and the structure becomes statically indeterminate. This design is therefore not to be recommended in the case of a new structure. The insertion of a central column for strengthening an old structure may, however, be well warranted by the circumstances of the case.

CHAPTER XVI

DEFORMATION OF TRUSSES. CAMBER

263. Displacement of Any Apex of a Truss. (Algebraic Method.) The elastic deformation of the members of a truss under the action of external forces causes a movement in space, or ' displacement ' of every point of the truss except a fixed point of support, the support being assumed for this purpose to be perfectly rigid. The component in any desired direction of the displacement of any given apex may be found by means of a simple general formula which may be derived as follows:

First consider the effect on the truss shown in Fig. 1, of a single force F exerted in any given direction. For convenience, assume that F is gradually increased from zero to a final value of 1 lb. Referring to Fig. (a), let Δ denote the component displacement in the direction of F, of the final displacement bb' $(=\Delta_a)$ of apex b, the other component, Δ', being measured from b in a direction at right angles to F. The work done by the force is then the product of the average force, $\frac{1}{2}$ lb., and the movement, Δ, of apex b, in the direction of the force. This product, $\frac{1}{2}\Delta$, is the work done by the force under the conditions stated, in producing deformation in the various truss members.

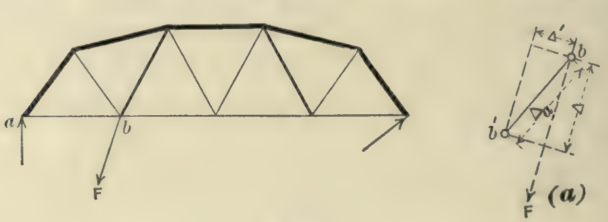


FIG. 1.

As F is gradually increased from zero to 1 lb., assume that the stress produced by F in any given member gradually increases from zero to u, and that the corresponding change of length in the member is l_1. The work done in deforming this member is $\frac{1}{2}ul_1$. Let δ denote the required component displacement of b in the direction of the force $F(=1 \text{ lb.})$ for the performance of this work, $\frac{1}{2}ul_1$. The work performed in this movement δ may then be expressed also as the product of δ and the average force, $\frac{1}{2}$ lb. Therefore $\frac{1}{2}\delta=\frac{1}{2}ul_1$, whence

$$\delta/l_1=u, \qquad (1)$$

and

$$\delta=l_1u. \qquad (2)$$

From Eq. (2) it is seen that the constant u is a factor—which may be called the *displacement factor*—by which the change of length, l_1, of the given member must be multiplied to obtain the component displacement, δ, in the direction F, due to that change of length. Although the displacement factor u is numerically equal to the stress in the given member from a force of 1 lb. applied to b in the direction in which the component displacement is to be found, it is in a more general sense the *ratio* of the stress, S, in the given member from *any* force F, applied to b as just described, to the force F. For since the stress varies proportionately to the force, the ratio $S/F=u/1=u$. Again, if F has any value other than unity, the quantities δ and l_1, in Eq. (1), become $F\delta$ and Fl_1, their ratio, u, remaining constant.

In applying Eq. (2) the cause to which the change of length l_1 is due is obviously immaterial. The change of length may be due to temperature, to play at pin-holes, to the action of loads anywhere on the bridge, etc. If the magnitude of l_1 is known, the component δ of the corresponding displacement of apex b may be found directly by Eq. (2). The component $\Delta(=\Sigma\delta)$ of the total displacement of apex b from simultaneous changes in length of any number of members, is found by the summation of the products l_1u for the individual members, whence

$$\Delta=\Sigma\, l_1u. \qquad (3)$$

This equation is perfectly general provided the changes of length l_1 are relatively small compared with the original lengths of the truss members, so that the effect of the deformation of the truss on the stresses u may be neglected without sensible error.

As the problem usually presents itself in practice, it is required to find the component in a given direction of the displacement of a given apex, from the elastic deformation of the truss by (1) the dead load, (2) some given live load, or (3) loadings (1) and (2) combined. If S denotes the total 'equivalent' dead-load stress (Art. 102) in a given member of length l, a the sectional area of the member, and E the modulus of elasticity of the material, $l_1 = Sl/aE$. For this case, Eq. (3) becomes

$$\Delta = \Sigma \frac{Sl}{aE} u. \qquad (4)$$

For steel E may be assumed at 29,000,000 lbs. per square inch.

264. Application of Algebraic Method. In applying Eq. (4), careful attention should be given to the signs of S and u. If both have the same sign, their product is plus, otherwise minus. The former indicates a movement in the direction under consideration, the latter a movement in the opposite direction. This may be seen by considering the derivation of Eq. (2). If u is a tensile stress, l_1 denotes a lengthening of the member in question, i.e., a *plus* change. Therefore a *lengthening* of that member, from any cause whatever, produces a component displacement of b along the line of action of F, in the *same* direction as the arrow-head. Conversely, if the same member is *shortened* from any cause whatever, the resulting component displacement of b along the line of action of F will be in a direction *opposite* to the arrow-head.

If one or both chords of the truss are inclined or the truss is otherwise irregular, the values of the stresses u, from a force of 1 lb. applied in the desired direction at the point whose component displacement in that direction is to be found, may be conveniently obtained graphically. If the stress u in any member is zero it shows that a change of length in that member will not affect Δ.

It is to be noted that a in Eq. (4) is the *gross* sectional area of the member, even in the case of riveted tension members, since the longitudinal deformation, not the strength, is under consideration.

Note. It is sometimes assumed that because the sectional area is reduced locally by rivet holes, the unit stress, and therefore the unit deformation, is somewhat increased. Therefore a correction is sometimes introduced by assigning a somewhat lower value to E (say 28,000,000) for riveted tension members. It may reasonably be assumed, however, that within the usual limits of working stresses the frictional grip between the heads of well-driven rivets and the parts connected will fully compensate for the reduction of section by rivet holes. Moreover, experience indicates that if gross sectional areas are used throughout in computing deflections, the actual deflections are apt to be less rather than greater than the computed ones. This may be attributed to the omission in deflection computations, of latticing and other details, as well as secondary stresses, all of which tend to decrease the primary stresses. It is better, therefore, to make no allowance for the effect of rivet holes in riveted tension members.

Displacements from Changes of Temperature. In applying Eq. (3) to the determination of component displacements due to changes of temperature, let t = number of degrees of change, c = coefficient of expansion and l = length of member. Then

$$l_1 = clt,$$

and

$$\Delta = ct\Sigma lu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For a rising temperature the values l_1 $(= clt)$ should be given a plus sign, and for a falling temperature, a minus sign.

Displacements from Play at Pin-holes. Again, to find the component displacements due to the play between the pins and pin-holes in a pin-connected truss, which usually amounts to 0.02 to 0.03 in. for each hole, assume a mean value of 0.025. Then in Eq. (3), $l_1 = 0.025$, this value being plus for tension members, and minus for compression members. Then with due regard to signs,

$$\Delta = \Sigma\, 0.025u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If a member is pin-connected at one end and riveted at the other, $l_1 = 0.0125$ for that member. Again, for compression chord members with one or both ends 'abutting' against the adjoining members or member, $l_1 = 0.0125$ and zero, respectively.

265. Example. Let it be required to find the vertical deflection at the center of a simple truss, from the dead load and full live loading—i.e., under loading which will develop the maximum stress in the middle chord members. The simultaneous live-load stresses S_l, throughout the truss may either be found accurately with the load system in the position for maximum stress in the middle chord members, or with sufficient accuracy for practical purposes by the following procedure. Find the intensity w_l of an 'equivalent' live load covering the entire bridge, which will produce a stress in the middle chord members equal to the maximum live-load stress in those members. Then if S_d is the dead-load stress in any member, and w_d is the uniform dead load per foot of bridge,

$$S=S_d+S_l=S_d\left(1+\frac{w_l}{w_d}\right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The component values δ, whose summation is the desired deflection, Δ, may then be found separately for each member and the results may be conveniently tabulated as follows:

Member.	l ins.	S by Eq. (7)	a sq.ins.	$\frac{Sl}{aE}$ (+) or (−)	u (+) or (−)	$\delta=\frac{Sl}{aE}u$ (+) or (−)
					$\Delta=\Sigma\frac{Sl}{aE}u=$	

Note. Instead of dividing Sl/a by E for each individual member, as implied in the heading of the fifth column, it is more convenient to drop the term E from the headings of the table, and to divide the final summation $\Sigma Slu/a$ by E at a single operation. If, however, it is desired to check the computed value of Δ by a graphic method to be described presently, the values Sl/aE must first be computed for each individual member, or else, if E is omitted, the scaled value of Δ must finally be divided by E.

The plus values in the last column denote downward, the minus values upward deflections at the center of the truss, and Δ represents their algebraic sum.

If Δ_a, the absolute movement in space of a given point of a truss subjected to given influences, is desired, the vertical and horizontal components Δ_v and Δ_h, of that movement may be found separately by determining the stresses u for a load of 1 lb. applied first, vertically, and second, horizontally, at the point in question. Finally

$$\Delta_a=(\Delta_v^2+\Delta_h^2)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

266. Check for Values of Displacement Factor. If one of the supports of the truss is free to roll horizontally, it is evident that no stresses will be developed by changes of temperature. Further, if all the truss members are made of the same material, the length of every member will be altered by an equal proportionate amount, and the new figure of the truss will be similar to its original figure. For a bridge with a horizontal lower chord and supports in the plane of that chord, the vertical deflections of the lower chord joints from a change of temperature are therefore zero, i.e., in Eq. (5), Art. 264,

$$\Sigma lu=0. \qquad \qquad (9)$$

Eq. (9) may be utilized as a check on the correctness of the values of the displacement factor, u, in the common case in which the deflections, for a given condition of loading, of the *lower chord joints* of a *through* bridge with a *horizontal* lower chord are desired.

In general, assume that the supports of the truss lie in the same horizontal plane, AB, Fig. 2, and that the movement at the roller end takes place in a horizontal direction. Then, if the displacement factors u are determined for the vertical deflection of a given joint, as, for example, K,

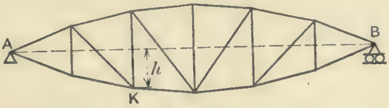


FIG. 2.

$$\Sigma lu=h, \qquad \qquad (10)$$

in which h denotes the vertical distance from K to AB. This may be proved as follows:

Evidently, under a change of temperature of t degrees, if c = coefficient of expansion, and Δ = component displacement of K in a vertical direction, $\Delta=ct\times h$. But from Eq. (5), $\Delta=ct\Sigma lu$. Hence, in general, $\Sigma lu=h$, as stated in Eq. (10).

For the conditions first considered, $h=0$. In the case of a bridge with horizontal chords and supports in the plane of the lower chord, if the vertical deflection of one of the upper chord joints is sought, h = height of truss. In applying Eq. (10) the values of u—regarded as stresses from a vertical load of 1 lb. at the joint whose deflection is to be found—must be given opposite signs for stresses opposite in character.

It is not necessary that the supports should lie in the same horizontal plane, provided the movement at the roller end takes place in a horizontal direction. If AB in Fig. 2 is inclined, and B moves horizontally, Eq. (10) is applicable, as before, to vertical deflections.

267. Displacements of the Apexes of a Truss (Graphic Method). It is sometimes desirable to ascertain the displacement of every apex of a truss, or the component displacements in a given direction of a certain number of apexes, from changes in length of one or more members. This may be more conveninetly accomplished graphically, as will be shown now, by the construction of a single Williot or displacement diagram than by applying the algebraic method (Art. 263) to each joint in turn.

268. Truss Fixed at One Support and Direction of One Member Unchanged. Let it be assumed at first that one support of the truss is fixed, and that the direction of one member is unaffected by the conditions which cause the displacements sought.

(*a*) *Triangular Frame.* In the triangular frame abc, Fig. 3(a), fixed at a and supported on rollers at b, let it be required to find the displacements of the apexes due to certain known changes in the length of the sides. In practice such changes are usually those due to elastic deformations under stress, variations in temperature, or play at pin-connected joints. Changes of length from such causes are so small in comparison with the total lengths of the members, that they must be represented in the figure to a greatly exaggerated scale.

Suppose that ab is shortened an amount bb'. Then b' is the new position of b and bb' is the displacement of that apex. Lay off cd equal and parallel to bb'. Suppose that $bc(=b'd)$ is lengthened an amount dc_1, and that ac is lengthened an amount cc_2. In the deformed frame the points c_1 and c_2 must coincide, the point of coincidence being the new position of c. This point may be located as the point of intersection of two arcs, the one described with a as a center and ac_2 as a radius, and the other with b' as a center and $b'c_1$ as a radius. Since the arcs are very short, however, in proportion to their radii, it is sufficiently accurate to draw the straight lines c_2c' and c_1c' perpendicular to the radii ac_2 and $b'c_1$ respectively. Their point of intersection, c', is then the new position of c, and cc' is the displacement of that apex.

It is implied in the foregoing that the changes of length bb', dc_1 and cc_2 are laid off to the same scale as the frame. In reality, however, these distances are so small in comparison with the lengths of the members that the construction just described becomes impracticable. This difficulty may be met, however, by constructing an independent displacement diagram to any desired scale.

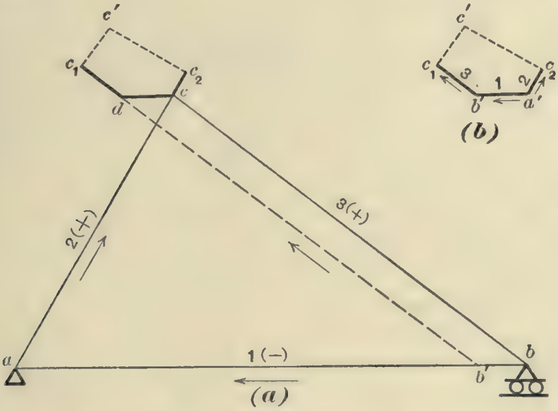


FIG. 3.

Thus, referring to Fig. (*b*), let a' denote a fixed reference point. Since member 1, Fig. (*a*), is shortened, b will move to the left, i.e., in the direction of the arrow. In Fig. (*b*) lay off $a'b'$ $(=bb')$ in the same direction from a'. Again, since member 2 is lengthened, c will move away from a, as shown by the direction of the arrow. In Fig. (*b*) lay off $2(=cc_2)$ in the same direction from a', and $3(=dc_1$, the plus deformation of member 3) in the right direction with respect to b'. From c_1 and c_2 draw lines perpendicular to $b'c_1$ and $a'c_2$, respectively, to their intersection at c'. The polygon in Fig. (*b*) is seen to be identical with that at the top of Fig. (*a*). The distances $a'b'$ and $a'c'$ are the displacements of apexes b and c respectively, which should be laid off in Fig. (*a*) in the directions of arrows pointing from a' to b' and c', respectively, in Fig. (*b*).

(*b*) *Multi-triangular Frame.* Assume the truss shown in Fig. 4(*a*) to be fixed at *a* and on rollers at *b*. Let it be further assumed that under the load *W* the members distinguished by plus and minus signs are in tension and compression, respectively, and that they are lengthened or shortened by known amounts represented to an enlarged scale in Fig. (*b*) by lines whose numbers correspond with the numbers of the members.

In Fig. (*a*) the movement of *b* is toward *a*, i.e., toward the right. Hence in Fig. (*b*) the deformation 1 of member 1 is laid off toward the right. Again, the movement of *c* with respect to *a* and *b* is in the directions of the arrows for *ac* and *bc*. Hence the deformations 2 and 3 in Fig. (*b*) are laid off in corresponding directions from *a*′ and *b*′. The intersection of the normals at the extremities of 2 and 3 locates *c*′. Similarly, 4 and 5 are laid off from *a*′ and *c*′ in the directions of the arrows on the corresponding members, and *d*′ is found as the point of intersection of the normals at the extremities of 4 and 5. Continuing in this manner, *e*′, *f*′, *g*′ and *h*′ are located in turn. The displacement of any apex as *f*, for example, is in the direction of *f*′ from *a*′ and its magnitude is *a*′*f*′.

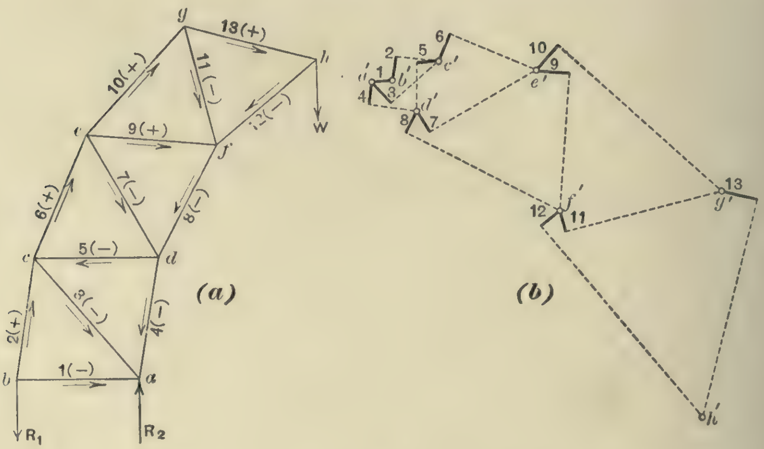


FIG. 4.

(*c*) *Simple Symmetrical Truss, Loads Counterbalanced.* The trusses in Figs. 5 (*a*) and 6 (*a*) are symmetrical, and it is required to find their displacements from known elastic deformations due to counterbalanced loads. In both trusses the left support is assumed to be fixed and the right support on rollers.

In Fig. 5 (*a*), the number of panels is even, so that for counterbalanced loading the middle post *ab* will remain vertical. In drawing the displacement diagram, Fig. (*b*), begin at a'. Since *ab* is in compression the displacement of *b* will be upward with respect to *a*. Hence this displacement, $a'b'$, is laid off upward from a' in Fig. (*b*). The positions of c', d', etc., in the displacement diagram are then found in alphabetic sequence in the manner previously described. Since the position of *g* in Fig. (*a*) is fixed, g' is the true point of reference in Fig. (*b*). The maximum vertical deflection, Δ_v, occurs at *a*. The greatest horizontal movement, Δ_e, in the left half of the truss, is that of apex *e*. The displacement diagram for the other half of the truss (not shown) would be symmetrical about $a'b'$ with the diagram in Fig. (*b*). The displacement of the right support is therefore $2\Delta_h$ toward the right.

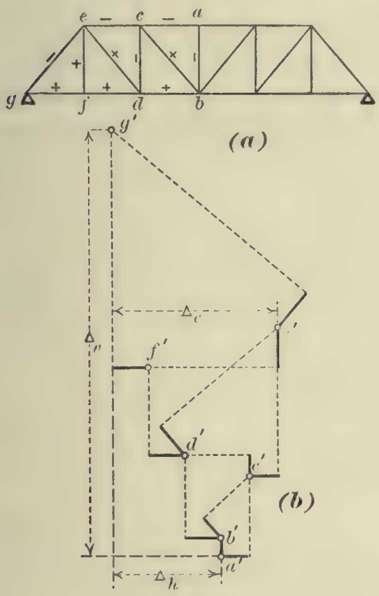


FIG. 5.

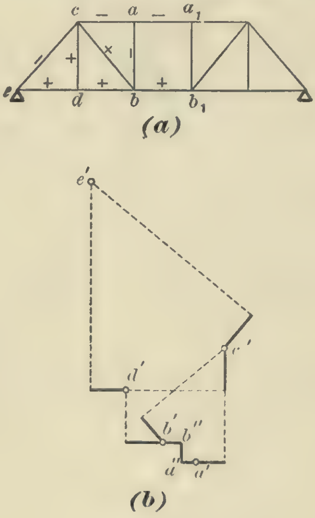


FIG. 6.

In Fig. 6 (*a*) the number of panels is odd, so that for counterbalanced loading the middle chord members, aa_1 and bb_1, remain horizontal. After deformation the horizontal displacement of a with respect to b will evidently equal half the sum of the shortening of aa_1 and the lengthening of bb_1. In Fig. (b), beginning at a', lay off $a'a''$ equal to half the shortening of aa_1, $a''b''$ equal to the shortening of ab, and $b''b'$ equal to half the shortening of bb_1. The displacement of b with respect to a as a fixed reference point is then correctly shown in Fig. (b) as to magnitude and direction, by the relative position of b' with respect to a'. The positions of c', d', and e' may then be established successively in the usual manner.

269. General Case. In the special case treated in the preceding article the position of one apex and the direction of one member are fixed. In the general case the apex at one of the supports is fixed, and the direction of the displacement of a second apex, (the other support in the case of a simple truss) is known; but the direction of every member is affected by the deformation of the truss.

Thus in the truss shown in Fig. 7, if the position of a is fixed and b is on rollers, it is known that the displacement of b will take place in a horizontal direction. For the general case of unsymmetrical loading, as with a load at c, for example, the direction of every member will be altered by the elastic deformation of the truss. The apex c, for example, will suffer a displacement cc' and the direction of ac will undergo an angular change α. If the displacement diagram is drawn on the assumption that the direction of ac remains unchanged and the resulting displacements are laid off in Fig. 7, apex b will appear to have suffered a displacement bb'', instead of its true displacement, bb', in a horizontal direction. The point b'' is the position which b' would occupy if the deformed truss were revolved around a through the angle α. The displacement diagram drawn as described will indicate the true *relative* displacement of every apex, that is to say, it will serve to determine the true form of the deformed truss; but in order to determine the *absolute* displacements the deformed truss must be revolved through the

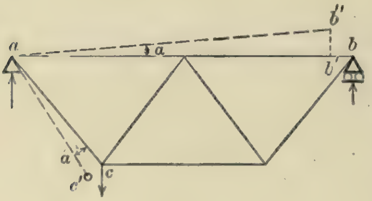


FIG. 7

angle α. Thus if the position of a and the direction of ac are assumed to be fixed, the relative movement of b will be bb''. The absolute movement, bb', of b is the resultant of the relative movement bb'' and the movement, $b''b'$, of b'' as the truss is revolved through an angle α around a. The absolute movement of every other apex may be found in precisely the same manner.

270. Displacements Due to Rotation. If a truss, as, for example, the one shown in Fig. 8 (*a*), is rotated through a small angle α about a fixed point a, the displacement of the apexes will obviously be proportional to their distances from the center of rotation, i.e.,

$$\Delta_1/r_1 = \Delta_2/r_2 = \Delta_3/r_3 = \Delta_4/r_4. \quad . \quad . \quad . \quad . \quad (11)$$

If the magnitude of any one of the displacements, Δ_1–Δ_4, is known, the magnitude of the others may be found by Eq. (11).

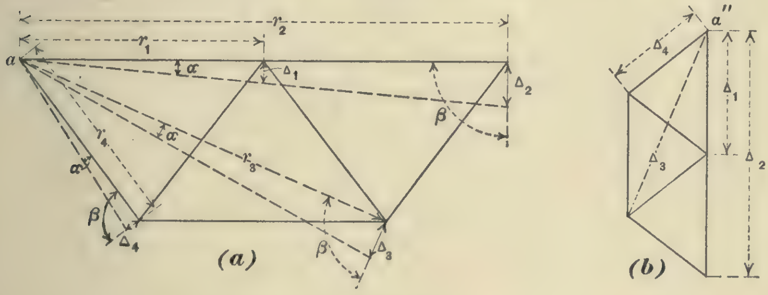
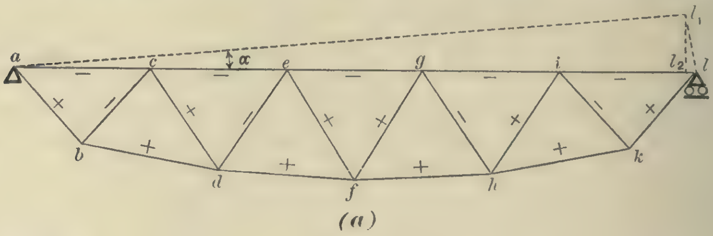


FIG. 8.

For the conditions under consideration the angle α is very small. It is sufficiently accurate, therefore, to assume that the displacements take place at right angles to r_1–r_4, i.e., to assume $\beta = 90°$.

If the magnitude of one of the displacements is known the other displacements may readily be found, both in magnitude and direction, by a simple graphical procedure. In Fig. (*b*) lay off the displacements Δ_1–Δ_4 from a'' in directions at right angles to the respective radii r_1–r_4. The point a'' in Fig. (*b*) corresponds to a in Fig. (*a*), and since the distances Δ_1–Δ_4 in Fig. (*b*), are respectively at right angles and proportional to the radii r_1–r_4 in Fig. (*a*), the lines joining the extremities of Δ_1–Δ_4 will form a figure similar to Fig. (*a*). The homologous sides in the two figures will be at right angles to each other.

This construction is admissible only if the angle α is very small, so that the assumption, $\beta = 90°$, may be made without sensible error. If the value of any one of the displacements Δ_1–Δ_4 is known, a second point in Fig. (*b*) may be established by laying off this displacement from a'' to a suitable scale. The figure may then be completed and the other displacements determined by the same scale.



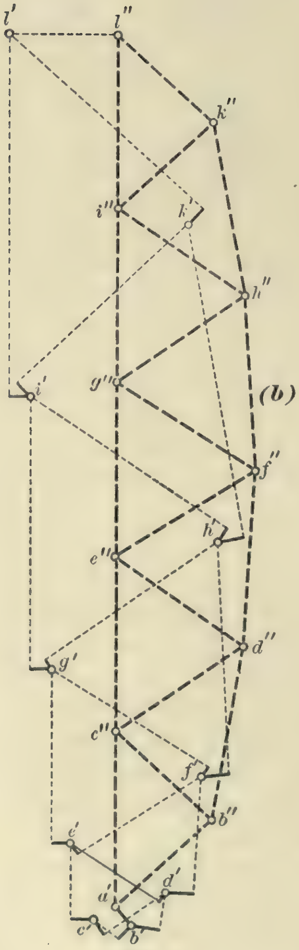


FIG. 9.

271. Example. General Case. Let it be required to find the displacement of every apex of the truss shown in Fig. 9, the end *a* being fixed and the other end on rollers. Let it be assumed that the stresses from which the deformations of the individual truss members are found were produced by unsymmetrical loading, so that the direction of every member, including the central chord member *eg*, is altered.

First draw the displacement diagram, Fig. (*b*), on the assumption that the direction of some member as *ab*, Fig. (*a*), for example, remains unchanged. Beginning at *a'* lay off the known deformation of *ab* in the proper direction from *a'* according to the slope of the member and the character, (+) or (−), of the deformation, as already explained. This locates the point b'. In the same way lay off from a' the deformation of ac, and from b' the deformation of bc. The point c' is then located by the intersection of perpendiculars from the extremities of these two lines. The points d', e', etc., and finally l', are established in a similar manner. The displacement of every apex for ab fixed in direction is now represented by the distance and direction of the corresponding point in the displacement diagram from a'. In Fig. (*a*) lay off $l\,l_1$ parallel to an imaginary line from a' to l' in Fig. (*b*), representing $a'l'$ to a reduced scale. The displacement of any other apex may be represented in the same way. It is evident that the true displacement of l must take place along a horizontal line through l. If the deformed truss is revolved around a through the angle $l_1al_2 = \alpha$, causing l_1 to move to l_2, the true displacement of any apex will be the resultant of the displacement already found, and its displacement in consequence of the rotation. For l, this resultant is $l\,l_2$, the movement being from l toward the left. Draw a vertical through a' in Fig. (*b*) until it intersects at l'' a horizontal through l'. The distances $a'l''$ and $l''l'$ in that figure then represent to an enlarged scale l_1l_2 and $l\,l_2$, respectively, in Fig. (*a*). The distance $a'l''$ is thus seen to represent the displacement of apex l by rotation. The displacements of the other apexes from this cause may therefore be found, as explained in the preceding article, by drawing the truss to scale on $a'l''$ as a base, as shown by the figure $a'b''c''d''e''$, etc.

The true displacements of the apexes, h, i, k and l, for example, are then the distances $h''h'$, $i''i'$, $k''k'$ and $l''l'$, respectively, measured in the directions indicated by the sequence in which the letters appear, i.e., $h''h'$ is in the direction from h'' to h', etc. For, considering apex h, for example, its displacement by rotation is $h''a'$ in the direction from h'' to a', whereas its relative displacement (for ab regarded as fixed in direction) is $a'h'$ in the direction from a' to h'. The resultant or true displacement of this apex is therefore $h''h'$ in the direction from h'' to h'. If only the vertical components of the displacements — i.e., the deflections—are wanted, as is frequently the case, a series of horizontal lines may be drawn through the points distinguished by letters in Fig. (*b*). The deflection of apex h, for example, is then the vertical intercept between the horizontals through h' and h''.

By laying off the deflections of successive apexes in either chord as ordinates from a horizontal base line and joining the extremities of these ordinates, *deflection curves* may be constructed which possess certain interesting and useful properties to be now considered.

272. Reciprocity of Deflections and Loads (Maxwell's Law). Assume the truss shown in Fig. 10 (*a*) to be loaded with a single weight of one pound at C, and let the vertical deflections of the upper chord apexes from this weight be found as just explained.

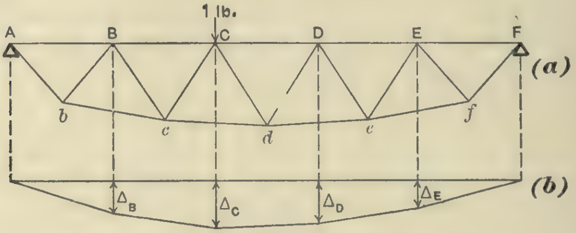


FIG. 10.

Let these deflections Δ_B, Δ_C, etc., be laid off from a horizontal base line as in Fig. (*b*), and let the deflection curve be drawn by joining the extremities of these ordinates. These deflections may also be computed by Eq. (4), Art. 263, viz:

$$\Delta = \Sigma \frac{Sl}{aE} u.$$

If this equation is used for determining Δ_B, for example, S denotes the stress in any member from the loading for which Δ_B is to be found, i.e., a load of one pound at C, and u denotes the stress in any member from a load of one pound applied at the point B at which the deflection is to be found.

Suppose now that the deflection at C is sought for the truss loaded with a single weight of one pound at B. Applying the above equation, S has now the same value as u in the previous case, and u has the same value as S in the previous case, so that the resulting values of Δ for the two cases are equal. That is to say, the deflection at B for a one-pound load at C is equal to the deflection at C for a one-pound load at B, or in general, *the deflection at any apex from a load applied at any other apex, is equal* *to the deflection at the second apex from an equal load applied at the first apex.*

273. Influence Line for Deflection. It has been seen that the deflection at C in Fig. 10 (a) from a load of one pound applied at any other apex is equal to the ordinate under that other apex in Fig. (b). The ordinates drawn to the deflection curve at the apexes of the loaded chord are therefore also the ordinates to the *influence line* for the deflection of apex C. Again, since the influence line, like the deflection curve, is a straight line between successive apexes of the loaded chord (Art. 191), the influence line for the deflection of any apex is the deflection curve of the loaded chord for a load of one pound placed at the apex in question. Hence the deflection of C from a load W applied anywhere along the loaded chord is found by multiplying the ordinate Δ in Fig. (b), under the load, by W. In general, if D denotes the deflection of the apex in question from any number of loads,

$$D = \Sigma W \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which the Δ's may be scaled from Fig. (b).

274. Camber. It is customary to assign such initial lengths to the members of a truss as to introduce a curvature in the truss as a whole. Referring to Fig. 11, the rise, c, at the center of the truss is called the *camber*. The aim is to provide such camber that when the truss is fully loaded the nominally horizontal chord or chords of the truss will be truly horizontal, i.e., the total deflection will then be equal to the camber. The deflection is due to the elastic deformation of the truss members from dead- and live-load stresses and, in the case of a pin-connected truss, to the 'play' between the pins and pin-holes. The influence of temperature changes on the deflection of a truss is usually disregarded, it being assumed that changes in temperature induce equal proportionate alterations of length in every member. The effects on the deflection of such alterations in the same direction in tension and compression members neutralize each other. Thus in the truss shown in Fig. 11, a lengthening of the lower chord will evidently cause the truss to deflect downward, whereas a lengthening of the upper chord will produce an opposite effect. The evaluation of Eq. (5), Art. 264, under the assumed conditions, for all members of the truss would lead to $\Delta = 0$.

The magnitude of the camber, as found by different methods to be considered presently, varies, but it amounts usually to from $\frac{1}{1900}$ to $\frac{1}{1200}$ of the span. If no camber is provided, the nominally horizontal chords of a truss will deflect below a horizontal line joining their extremities, even under the action of the dead load alone. This is considered objectionable on the ground of appearance, although the effect is so slight that it would probably be noticeable only to an observer sighting along the axis of the chord. Aside from appearance camber is provided in order that when the truss is fully loaded the successive compression chord members will assume the relative angular positions toward each other for which the abutting ends of these members were faced, so that for this condition of maximum chord stress the compression chord as a whole will be nearly or quite free from bending. If, for example, the compressive chord is nominally horizontal, the ends of the chord members are faced in planes at right angles to their longitudinal axes, and if proper allowance is made for camber, these chord members will be horizontal when the truss is fully loaded. If, on the other hand, no allowance is made for camber, it would be necessary to face the ends of the compression chord members in planes making such angles with their longitudinal axes that the ends will come in contact throughout when the truss is fully loaded. Although the splice plates connecting the ends of adjacent chord members will prevent actual angular movement at the joints under variations of loading, bending moments (and therefore eccentric stresses in the chords) are developed under conditions of loading for which the ends of the chord members are held under constraint by the splice plates. The aim is to reduce these bending moments to zero, in the manner above indicated, when the truss is fully loaded and the chord stresses are maximum.

275. Camber Computations by Approximate Methods. *First Method.* By this method the camber is assumed to be a function of the span only, the ratio usually chosen for c/l being $\frac{1}{1200}$. Considering the arc of curvature, Fig. 11, to be that of a circle of radius R,

$$(2R-c)c=\tfrac{1}{4}l^2.$$

Since the arc is very flat this equation becomes without sensible error

$$2Rc=\tfrac{1}{4}l^2$$

whence

$$c=\frac{l^2}{8R}.$$

Drawing BC parallel to the radius at A, angle $CBD(=\alpha)$ is equal to the angle between the radii at A and B. Since α is very small, assume that $BC=BD$. Then from similar triangles, $l/R=i/h$, whence

$$c=\frac{il}{8h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Letting $c=l/1200$,

$$i=\frac{h}{150}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

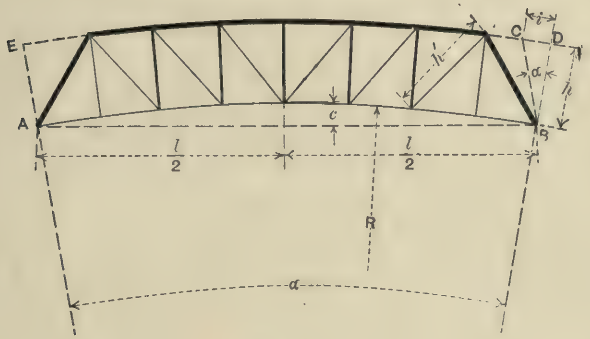


FIG. 11.

in which i represents the difference in the lengths of the arcs AB and DE. If N denotes the number of panels in the truss (in this case, 8) and i_p the difference between the panel lengths of the upper and lower chords,

$$i_p=\frac{h}{150N}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The desired camber may therefore be introduced in the truss by shortening each lower panel, or lengthening each upper panel, by the amount i_p. In order that the span length, l, assumed in the computations, which is usually some even number of feet, may not be altered, it is preferable to lengthen the upper chord. For $N=8$ and $h=32$ ft., by Eq. (15), $i_p=0.32$ in.

The lengths, h', of the diagonals and inclined end posts must be computed as the diagonals of a trapezoid. Thus if p denotes the original panel length,

$$h' = [(p + \tfrac{1}{2} i_p)^2 + h^2]^{\frac{1}{2}}. \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

By this method the lengths of the verticals are unaffected.

Second Method. By this method the upper chord is made $\frac{1}{8}$ in. longer than the lower chord for each 10 ft. of length, i.e.,

$$i = \frac{l}{960}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

By Eq. (13), the corresponding camber will be

$$c = \frac{l}{7680\,h}. \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

This method is more rational than the first method in that the camber, c, by Eq. (18), varies inversely as the depth of truss, which is true also of the deflection, for equal spans and unit stresses. The ratio l/h usually ranges from 4 to 6, whence c by Eq. (18) ranges from about $\frac{1}{1900}l$ to $\frac{1}{1300}l$, as compared with the value $\frac{1}{1200}l$ used in connection with the first method. For $c = \frac{1}{1600}l$, Eq. (13) becomes

$$i = \frac{h}{200}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

for which the camber will be in fair average agreement with that by Eq. (18). The values i by Eqs. (17) and (19) are equal for $l/h = 4.8$.

276. Camber Computations by Exact Method. By this method the aim is to assign such an initial length to each member that when the truss is fully loaded each member will be of the length assumed in the determination of the stresses. Calling the latter length the 'hypothetical' length of the member, between centers of end joints,

Let l = hypothetical length of a given member;
l_i = its initial length;
S = its total stress for bridge fully loaded;
a = its sectional area;
E = modulus of elasticity of the material.

Then for a tension member,

$$l_i = l\left(1 - \frac{S}{aE}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and for a compression member,

$$l_i = l\left(1 + \frac{S}{aE}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In these equations S is the sum of the dead-load stress and the live-load stress which exists in the member when the bridge is fully loaded. It is not customary in calculations for camber to increase the live-load stresses by allowances for impact, partly because the correct allowances cannot be accurately fixed, and partly because the effect of impact prevails only momentarily, the stresses oscillating between limits of which the computed stresses may be regarded as mean values. Again, a is the *gross* sectional area of the member, even for riveted tension members, for reasons explained in Art. 264.

For pin-connected members allowance should be made for a difference of 0.025 in. between the diameter of the pin and the pin-hole, by deducting this amount from the lengths of tension members obtained by Eq. (20), and adding it to the lengths of compression members obtained by Eq. (21). No allowance should be made, however, in the case of compression members abutting against adjacent members at both ends.

The simultaneous live-load stresses, S_l, for full loading may be found with sufficient accuracy for practical purposes in the manner explained in Art. 265, and the values of S in Eqs. (20) and (21) may then by determined by Eq. (7) of that article.

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